${}_{T_1}[L|_{W_1}]_{T_1} = J(\lambda, n_1)$ where $n_1 = m$.

Now look at the quotient space V/W_1 and repeat the process using $\bar{L} : V/W_1 \to V/W_1$ in place of L, and using what we know about $\mathrm{Char}_{\bar{L}}(t)$ and $m_{\bar{L}}(t)$. Arguing by induction on $\dim(V)$, we get a basis $\bar{T}$ of V/W_1 satisfying the theorem for $\bar{L}$ so just need to lift basis cosets of V/W_1 to basis vectors of V that extend T_1 to a J-basis of V. □

In practical problems it is only reasonable to do small examples.

Examples. Suppose $L: V \to V$ has $\mathrm{char}_L(t) = (t-1)^5(t-2)^3$ and $m_L(t) = (t-1)^3(t-2)^2$.
Then we find all possible J-forms rep'ing L.
$J = \mathrm{diag}(J_1, J_2) \in F^8_8$ where $J_1 \in F^5_5$, $J_2 \in F^3_3$ are Jordan blocks for $\lambda_1 = 1$ and $\lambda_2 = 2$, resp.
The largest J-block in J_1 must be $J(1,3)$ since $m_1 = 3$
so the remaining J-blocks in J_1 can be either $J(1,2)$ or two of $J(1,1) = [1]$. $J_1 = \mathrm{diag}(J(1,3), J(1,2))$ or $(J(1,3), [1], [1])$
The largest J-block in J_2 must be $J(2,2)$ since $m_2 = 2$
so the remaining J-block in J_2 must be $J(2,1) = [2]$.
Taken together, $J = (J(1,3), J(1,2), J(2,2), [2])$ where $g_1 = 2$ and $g_2 = 2$, or $J = (J(1,3), [1], [1], J(2,2), [2])$ where $g_1 = 3$ and $g_2 = 2$.

Note: $k_1 = 5$ and $m_1 = 3$ meant the options for J_1 corresponded to the two partitions of 5 with largest part 3, that is $5 = 3+2$ or $5 = 3+1+1$.

$k_2 = 3$, $m_2 = 2$ meant the options for J_2 corresponded to the partition of 3 with largest part 2, that is, $3 = 2+1$.

If, instead $Char_2(t) = (t-1)^5(t-2)^3$, $m_2(t) = (t-1)^2(t-2)$ then the options for J_1 would correspond to the two partitions of 5 with largest part 2, that is, $5 = 2+2+1$ or $2+1+1+1$, so $J_1 = diag(J(1,2), J(1,2), [1])$, $g_1 = 3$, or $J_1 = diag(J(1,2), [1], [1], [1])$, $g_1 = 4$.

For J_2 get only $diag([2],[2],[2])$, $g_2 = 3$, corresp. to partit. $3 = 1+1+1$ with largest part 1.

If $\mathrm{Char}_L(t) = (t-1)^5(t-2)^4$ and $m_L(t) = (t-1)^3(t-2)^2$ 252

then J_1 is either $\mathrm{diag}(J(1,3), J(1,2)) = A$ or $\mathrm{diag}(J(1,3), [1], [1]) = B$

and J_2 is either $\mathrm{diag}(J(2,2), J(2,2)) = C$ or $\mathrm{diag}(J(2,2), [2], [2]) = D$

So there are 4 options for J:

① $J = \mathrm{diag}(A, C)$ with $g_1 = 2$ and $g_2 = 2$

② $J = \mathrm{diag}(A, D)$ with $g_1 = 2$ and $g_2 = 3$

③ $J = \mathrm{diag}(B, C)$ with $g_1 = 3$ and $g_2 = 2$

④ $J = \mathrm{diag}(B, D)$ with $g_1 = 3$ and $g_2 = 3$

Warning: The geom. mults. do not always uniquely determine what J is.

Example: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^3_3$, so $\mathrm{Char}_A(t) = (t-1)^3$

Find Jordan form J similar to A and the invertible $P \in \mathbb{R}^3_3$ such that $J = P^{-1}AP$.

$$A - 1I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = N \text{ and } N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $m_A(t) = (t-1)^3$, so $J = J(1,3) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$A_1 = A_1^{(1)} = Nul(A - 1I_3) = \left\{ \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\} \text{ since } \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r.r.}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} x_1 = r \in \mathbb{R} \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \qquad g_1 = 1.$$

is RREF:

$$A_1^{(2)} = Nul(N^2) = \left\{ \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r, s \in \mathbb{R} \right\} \text{ since } \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \left.\begin{matrix} x_1 = r \\ x_2 = s \end{matrix}\right\} \text{free} \\ x_3 = 0 \end{matrix}$$

$A_1^{(3)} = Nul(N^3) = \mathbb{R}^3$ since $N^3 = 0_3^3$.

To get a J-basis $T = \{v_1, v_2, v_3\}$, first pick $\theta \neq v_3 \in A_1^{(3)} - A_1^{(2)}$. Any vector in $\mathbb{R}^3$ not in $\left\{\begin{bmatrix} r \\ s \\ 0 \end{bmatrix}\right\}$. I would pick $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then

$$v_2 = (A - 1I_3)v_3 = Nv_3 = Col_3(N) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in A_1^{(2)} - A_1^{(1)}$$

and $v_1 = (A - 1I_3)v_2 = Nv_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in A_1^{(1)}$.

$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $P = {}_S P_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To check $P^{-1}AP = J$, look at $AP = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and at

$PJ = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = AP$ checks.

Th. Let $L:V\to V$ and $f(t)\in F[t]$. Then 12 55
$\operatorname{Ker}(f(L)) \leq V$ is an L-invariant subspace of V.

Pf. Let $W=\operatorname{Ker}(f(L))=\{v\in V \mid f(L)v=\theta\}$. Then
$\theta\in W$ since $f(L)\theta=\theta$, and if L is linear, so
is $f(L):V\to V$ as a lin. comb. of powers of L.
Ker of any lin. map is a subspace. Why is W L-invar?
Let $v\in W$, then $L(v)\in W$ since $f(L)(L(v))=$
$(f(L)\circ L)(v)=(L\circ f(L))(v)=L(f(L)v)=L(\theta)=\theta$.
We used that L commutes with powers of L. $\square$

Suppose $L:V\to V$ and $\mathrm{Char}_L(t)=\prod_{i=1}^{r} f_i(t)^{k_i}$ for
distinct irreducible $f_i(t)\in F[t]$.
If $F=\mathbb{R}$, $1\leq \deg(f_i(t))\leq 2$. Let $W_i=\operatorname{Ker}\left(f_i(L)^{k_i}\right)$.