<u>Th</u>: If $m_L(t) = \prod_{i=1}^{r} f_i(t)^{m_i}$ then $W_i = \ker(f_i(L)^{m_i})$ and $V = W_1 \oplus \cdots \oplus W_r$ and if $L_i = L|_{W_i}$ then $m_{L_i}(t) = f_i(t)^{m_i}$.
(this is called the Primary Decomposition Theorem.)

<u>pf.</u> This follows from two next Theorems, whose proofs are in the textbook, and which follow ideas already given in the Jordan Form proof.

<u>Th</u>: Let $L: V \to V$ and $f(t) = g(t)h(t) \in F[t]$ with g.c.d.$(g(t), h(t)) = 1$ and $f(L) = 0_V^V$. Then $U = \ker(g(L))$ and $W = \ker(h(L))$ are L-invar. subspaces of V, and $V = U \oplus W$.

Th: With notation as in the last Theorem, if $f(t) = m_L(t)$ and $g(t), h(t)$ are monic, then $L|_U$ has min. poly. $g(t)$ and $L|_W$ has min. poly. $h(t)$.

Cyclic Subspaces. Let $L : V \to V$, $\dim(V) < \infty$.

For $\theta \neq v \in V$ define $Z(v, L) = \{f(L)v \in V \mid f(t) \in F[t]\}$ the L-cyclic subspace of V generated by v, and let $L_v = L|_{Z(v,L)}$. To get another viewpoint, consider the set $\{v, L(v), L^2(v), L^3(v), \ldots\}$. Let $1 \leq k \in \mathbb{Z}$ be the *least* s.t. $\{v, L(v), \ldots, L^k(v)\}$ is dep. so $k \leq \dim(V)$ and $\{v, L(v), \ldots, L^{k-1}(v)\}$ is indep. Can find $a_i \in F$ s.t. $\theta = \sum_{i=0}^{k} a_i L^i(v)$ with $a_k = 1$.

This says $L^k(v) = -\sum_{i=0}^{k-1} a_i L^i(v) \in \langle v, L(v), \dots, L^{k-1}(v) \rangle$

and for poly $f(t) = \sum_{i=0}^{k} a_i t^i$, $f(L)(v) = \theta$

so $f(L)(L^i(v)) = L^i(f(L)(v)) = L^i(\theta) = \theta$.

But no smaller degree $g(t)$ does $g(L)(v) = \theta$
by indep. of $\{v, \dots, L^{k-1}(v)\}$ (min. choice of k).

Thus, $f(t) = m_{L,v}(t)$. Also,

$$L^{k+1}(v) = -\sum_{i=0}^{k-1} a_i L^{i+1}(v) \in \langle L(v), \dots, L^{k-1}(v), L^k(v) \rangle$$
$$\leq \langle v, L(v), \dots, L^{k-1}(v) \rangle$$

and $\langle v, L(v), \dots, L^{k-1}(v) \rangle$ is L-invariant so invar.
under all polys $g(L)$ for any $g(t) \in F[t]$.

Th: Let $Z(v,L)$, L_v, $m_v(t)=m_{L_v}(t)$ be as defined above. Then

(1) $B=\{v, L(v), \dots, L^{k-1}(v)\}$ is a basis of $Z(v,L)$, so

$$\dim(Z(v,L))=k$$

(2) Min. poly. of L_v is $m_v(t)=\sum_{i=0}^{k} a_i t^i$

(3) $$ {}_B[L_v]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix} = C(m_v(t)) $$

is called the <u>companion</u> matrix of monic poly. $m_v(t)$. This matrix also has char. matrix $m_v(t)$.

Examples: Find the companion matrix for the poly. $f(t) = (t^2+1)^2$. L260

Solution: $f(t) = t^4 + 2t^2 + 1 = t^4 + 0t^3 + 2t^2 + 0t + 1$

so $C((t^2+1)^2) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Exercise: Check that $f(t)$ is the char. and min. poly of this matrix.

Note: $C(t^2+1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ since $t^2+1 = t^2+0t+1$

Question: If $A = \text{diag}\left(C((t^2+1)^2), C(t^2+1)\right)$ then what is $\text{char}_A(t)$? what is $m_A(t)$?

Answer: $\text{char}_A(t) = (t^2+1)^2(t^2+1) = (t^2+1)^3$

$m_A(t) = (t^2+1)^2 = \text{l.c.m.}\left((t^2+1)^2, t^2+1\right)$

Ex: Let $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, $L = L_A : \mathbb{R}^2 \to \mathbb{R}^2$.

$char_L(t) = char_A(t) = \det(tI_2 - A) = \det \begin{bmatrix} t+1 & 1 \\ -1 & t \end{bmatrix} = t^2+t+1$

which is irreducible over $\mathbb{R}$, so A is not diag-able over $\mathbb{R}$. Can't do Jordan form over $\mathbb{R}$ since $char_A(t)$ is not a product of linear factors.

Let $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2$. Find $Z(v, L) = \langle v, L(v), \ldots \rangle$

$L(v) = Av = Col_1(A) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $T = \left\{ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, L(v) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

is indep. and spans $\mathbb{R}^2$, and

$L^2(v) = L(L(v)) = A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and

$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ has solution: $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right]$

$x_1 = -1$ so $L^2(v) = -L(v) - v$ so $(L^2 + L + I)(v) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = -1$ so L satisfies $p(t) = t^2 + t + 1$.

This is no suprise since $p(A) = \mathrm{char}_A(A) = 0_2^2$ /262

by Cayley-Hamilton Th. But what is ${}_T[L]_T$?

If $T = \left\{ v_1 = v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = L(v) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ then

$L(T)$ is: $L(v_1) = v_2$ and $L(v_2) = L^2(v_1) = -v_1 - v_2$

so $[L(v_1)]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $[L(v_2)] = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ so

$${}_T[L]_T = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = C(t^2+t+1) = P^{-1}AP \text{ where } P = {}_SP_T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

<u>Ex</u>: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $L = L_A : \mathbb{R}^3 \to \mathbb{R}^3$. Let $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

Find $Z(v, L)$. Can we use it to get a companion matrix similar to A?