Look at $Z(v,L) = \langle v, L(v), L^2(v), \dots \rangle$.

$v = \begin{bmatrix}1\\0\\0\end{bmatrix}$ so $L(v) = Av = Col_1(A) = \begin{bmatrix}1\\0\\0\end{bmatrix} = v_1$ so

$Z(v,L) = \langle v \rangle = \left\{ \begin{bmatrix}r\\0\\0\end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\} \subsetneq \mathbb{R}^3$.

Try a different $v = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Then $L(v) = Av = \begin{bmatrix}1\\1\\0\end{bmatrix}$

so $\left\{ \begin{bmatrix}0\\1\\0\end{bmatrix} = v, \begin{bmatrix}1\\1\\0\end{bmatrix} = L(v) \right\}$ is indep.

$L^2(v) = L\begin{bmatrix}1\\1\\0\end{bmatrix} = A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and

$\left\{ \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$ is dep. $-\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}$ so

still <u>don't</u> get $Z(v,L) = \mathbb{R}^3$.

Try $v = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $Av = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $A^2v = A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ and

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$$\left\{ v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ v_2 = Av_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ v_3 = Av_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \text{ is}$$

indep. spans $\mathbb{R}^3$, $Z(v, L) = \mathbb{R}^3$.

$$A v_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} \in \langle v_1, v_2, v_3 \rangle \quad \text{Solve } \sum_{i=1}^{3} x_i v_i = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{aligned} x_1 &= 1 \\ x_2 &= -3 \\ x_3 &= 3 \end{aligned} \quad \text{says}$$

$0\ -1\ -2\ -3$ $\qquad$ $0\ 0\ -2\ -6$

$$A v_3 = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{v_1} - 3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{Av_1} + 3 \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{A^2 v_1}$$

$= A^3 v_1$

so $(A^3 - 3A^2 + 3A - I_3)\, v_1 = 0^3_1$ so

$(\sim\sim\sim)\, A^i v_1 = 0^3_1$ for $i = 0, 1, 2$

$$\underbrace{(A^3-3A^2+3A-I_3)}_{=O^2_2}\ \underbrace{Z(v,L)}_{=\mathbb{R}^3} = O^3_1$$

poly $p(t)=t^3-3t^2+3t-1 = \mathrm{char}_A(t)$ is satisfied by A (of course!) If $T=\left\{v_1=\begin{bmatrix}0\\0\\1\end{bmatrix}, v_2=\begin{bmatrix}1\\1\\1\end{bmatrix}, v_3=\begin{bmatrix}3\\2\\1\end{bmatrix}\right\}$

then $Av_1=v_2$, $Av_2=v_3$, $Av_3=v_1-3v_2+3v_3$

so $${}_T[L]_T=\begin{bmatrix}0&0&1\\1&0&-3\\0&1&3\end{bmatrix}=C(t^3-3t^2+3t-1)=C((t-1)^3)$$

$$=P^{-1}AP \text{ for } P={}_SP_T=\begin{bmatrix}0&1&3\\0&1&2\\1&1&1\end{bmatrix}.$$ Check!

Pf. Let $f(t) = \sum_{i=0}^{n} a_i t^i$ with $a_n = 1$ and let $C = C(f(t))$ be the companion matrix of $f(t)$. Then

$$\mathrm{Char}_C(t) = \det(tI_n - C) = \det \begin{bmatrix} t & & & & a_0 \\ -1 & t & & 0 & a_1 \\ & -1 & \ddots & & \vdots \\ & & \ddots & t & a_{n-2} \\ 0 & & & -1 & t+a_{n-1} \end{bmatrix} =$$

$$t \det \begin{bmatrix} t & & & & a_1 \\ -1 & t & & 0 & a_2 \\ & -1 & \ddots & & \vdots \\ & & \ddots & t & a_{n-2} \\ 0 & & & -1 & t+a_{n-1} \end{bmatrix}$$ (Cofactor expansion along row 1)

$$+(-1)^{n+1} a_0 \det \begin{bmatrix} -1 & t & & 0 \\ & -1 & t & \\ & & \ddots & t \\ 0 & & & -1 \end{bmatrix} = t\,\mathrm{Char}_D(t) + a_0$$ where

$D = C\left(\sum_{i=0}^{n-1} a_{i+1} t^i\right)$. By induction on n, we have

$\mathrm{Char}_D(t) = \sum_{i=0}^{n-1} a_{i+1} t^i$ so $\mathrm{Char}_C(t) = t \sum_{i=0}^{n-1} a_{i+1} t^i + a_0$

$= \sum_{i=0}^{n-1} a_{i+1} t^{i+1} + a_0 = f(t)$. The base case $n=2$ is easy,

$$\det\begin{bmatrix} t & a_0 \\ -1 & t+a_1 \end{bmatrix} = t(t+a_1)+a_0 = t^2+a_1t+a_0.$$

Also want to show that $m_C(t) = f(t)$.
Let $L = L_C : F^n \to F^n$ so ${}_S[L]_S = C$ for std. basis S.
Then $L(e_j) = e_{j+1}$ for $1 \le j \le n-1$ but $L(e_n) = -\sum_{i=1}^{n} a_{i-1} e_i$
from the columns of C $\left(\mathrm{Col}_j(C) = [L(e_j)]_S\right)$. We
know $\{e_1, e_2 = L(e_1), e_3 = L^2(e_1), \dots, e_n = L^{n-1}(e_1)\} = S$ is indep.
so if $m_C(t) = \sum_{i=0}^{r} b_i t^i$ for $r < n$ then $\theta =$
$m_C(L)e_1 = \sum_{i=0}^{r} b_i L^i(e_1)$ would be a dep. relation on
S, so all $b_i = 0$ contradicting $m_C(t)$ monic.
So $\deg(m_C(t)) = n = \deg(f(t))$ and $m_C(t)$ divides
$f(t)$ and both monic forces $m_C(t) = f(t)$. $\square$

What does the Primary Decomp. Th. tell us in the case $r=1$? $m_L(t) = f_1(t)^{m_1}$ for irred. poly $f_1(t)$, and say $\deg(f_1(t)) = d_1$ so $n = d_1 K_1$ where $Char_L(t) = f_1(t)^{K_1}$. $V = W_1 = \ker(f_1(L)^{m_1})$.
$\exists\, 0 \neq v_1 \in V$ s.t. $\dim(Z(v_1, L))$ is maximal. We know $Z(v_1, L)$ is an L-invar. subspace and $L_{v_1} = L|_{Z(v_1,L)}$ has $Char_{L_{v_1}}(t) = m_{L_{v_1}}(t)$, $C(m_{L_{v_1}}(t)) = [L_{v_1}]_{T_1}^{T_1}$ for basis $T_1 = \{v_1, L(v_1), \ldots, L^{e_1 - 1}(v_1)\}$ of $Z(v_1, L)$.
Since $m_{L_{v_1}}(t)$ divides $m_L(t) = f_1(t)^{m_1}$ we have $m_{L_{v_1}}(t) = f_1(t)^{p_1}$ for $p_1 \leq m_1$, $\deg(m_{L_{v_1}}(t)) = d_1 p_1$
$e_1 = \dim(Z(v_1, L))$ was maximal.

If $Z(v_1, L) = V$ then $m_L(t) = m_{L_{v_1}}(t) = \mathrm{Char}_L(t)$ has degree $n = d_1 k_1$ and $m_1 = k_1$ so

$${}_{T_1}[L]_{T_1} = C(m_L(t)) = C\left(f_1(t)^{m_1}\right) = C\left(f_1(t)^{k_1}\right)$$

is a single companion matrix.

But if $Z(v_1, L) \subsetneqq V$ then look at $V/Z(v_1, L)$ and the induced $\bar{L}$ on it. Find $v_2 \in V$, $v_2 \notin Z(v_1, L)$, so $\bar{v}_2 \neq \theta_{V/Z}$ and where $Z(\bar{v}_2, \bar{L})$ has maximal dim. Get basis $\bar{T}_2$ of $Z(\bar{v}_2, \bar{L})$ s.t. ${}_{\bar{T}_2}[\bar{L}]_{\bar{T}_2} = C(m_{\bar{L}_{\bar{v}_2}}(t))$ is companion matrix of min. poly. for $\bar{L}_{\bar{v}_2} = \bar{L}|_{Z(\bar{v}_2, \bar{L})}$ and play same game again.

Use connection between V and $V/Z(v_1, L)$ $\lfloor 270$
to get basis $T_1 \cup T_2$ for $Z(v_1, L) \oplus Z(v_2, L) \leq V$
and block diag. form for L restricted to it.
If that is not all of L, do another quotient.
Eventually (or by induction) can find
$Z(v_1, L) \oplus Z(v_2, L) \oplus \cdots \oplus Z(v_r, L) = V$ where
dimensions are non-increasing by choices
of $v_1, \ldots, v_r$. Each $Z(v_i, L)$ has char. poly.
$=$ min. poly. $= m_i(t) = f_1(t)^{p_i}$ of degree $d_1 p_i$

so $\mathrm{Char}_L(t) = \prod_{i=1}^{r} f_1(t)^{p_i} = f_1(t)^{k_1}$ and $p_1 + \cdots + p_r = k_1$

But $m_L(t) = \text{l.c.m.}\left(f_1(t)^{p_1}, \ldots, f_1(t)^{p_r}\right) = f_1(t)^{p_1}$

so $p_1 = m_1$.

In summary, if $\mathrm{Char}_L(t) = f_1(t)^{k_1}$ and $m_L(t) = f_1(t)^{m_1}$ for irred $f_1(t)$ of degree d_1 then $\exists\, v_1, \dots, v_r \in V$ s.t. $V = Z(v_1, L) \oplus \cdots \oplus Z(v_r, L)$ with $\dim(Z(v_1, L)) \geq \cdots \geq \dim(Z(v_r, L))$,
$L_{v_i} = L|_{Z(v_i, L)}$ has $\mathrm{Char}_{L_{v_i}}(t) = m_{L_{v_i}}(t) = f_1(t)^{p_i}$
$\dim(Z(v_i, L)) = d_1 p_i$, $p_1 + \cdots + p_r = k_1$, $p_1 = m_1$.
There is a basis T_i for $Z(v_i, L)$ made by
$T_i = \{v_i, L(v_i), \dots, L^{d_1 p_i - 1}(v_i)\}$, $T = T_1 \cup \cdots \cup T_r$ is a basis of V and ${}_T[L]_T =$
$\mathrm{diag}\left(C(f_1(t)^{p_1}), C(f_1(t)^{p_2}), \dots, C(f_1(t)^{p_r})\right)$ and

$$p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$$

Note: This is similar to the Jordan form, but basic blocks are companion matrices. Given k_1 and m_1 the number of options for ${}_T[L]_T$ are the number of partitions of $k_1 = p_1 + \cdots + p_r$ into parts with largest part $p_1 = m_1$.

Call this block diagonal form rep'ing L the Rational Canonical Form (RCF) of L.

EX. Suppose $\mathrm{Char}_L(t) = (t^2+1)^5$ and $m_L(t) = (t^2+1)^2$

Find all possible RCF matrices that represent L. Here, $L: V \to V$ over $F = \mathbb{R}$ so t^2+1 is irred.

Solution. $k_1 = 5$, $m_1 = 2$, $\dim(V) = 10$.

$p_1 = m_1 = 2$ so the possible partitions of 5 with largest part 2 are $5 = 2+2+1$ and $5 = 2+1+1+1$.

There are only two possible RCF's for L:

① $\mathrm{diag}(C((t^2+1)^2), C((t^2+1)^2), C(t^2+1))$

② $\mathrm{diag}(C((t^2+1)^2), C(t^2+1), C(t^2+1), C(t^2+1))$.

These companion matrices are on page 260.

A shorter way to write this is:

Let $C_1 = C((t^2+1)^2)$, $C_2 = C(t^2+1)$ so

① $\mathrm{diag}(C_1, C_1, C_2)$ or ② $\mathrm{diag}(C_1, C_2, C_2, C_2)$

are the two options.

General Case: $\mathrm{Char}_L(t) = \prod_{i=1}^{r} f_i(t)^{K_i}$ and $m_L(t) = \prod_{i=1}^{r} f_i(t)^{m_i}$ with $\deg(f_i(t)) = d_i$ so $\dim(V) = n = \sum_{i=1}^{r} d_i K_i$. For $1 \le i \le r$, $W_i = \ker(f_i(L)^{m_i})$ has a direct sum decomposition into cyclic subspaces. $V = W_1 \oplus \cdots \oplus W_r$ and $W_i = Z(v_{i1}, L) \oplus \cdots \oplus Z(v_{ir_i}, L)$. The RCF of L is a block diag. form $R = \mathrm{diag}(C_1, \ldots, C_r)$ where each $C_i = \mathrm{diag}(C_{i1}, \ldots, C_{ir_i})$ and each C_{ij} is a companion matrix for $Z(v_{ij}, L)$.