Ex. Let $L: V \to V$ over $\mathbb{R}$ have
$\mathrm{Char}_L(t) = (t^2+1)^5(t^2+t+1)^4$ and
$m_L(t) = (t^2+1)^3(t^2+t+1)^2$. Find all possible
RCF's of L.

For $f_1(t) = t^2+1$, $k_1 = 5$, $m_1 = 3$, partitions are
$5 = 3+2$ or $5 = 3+1+1$, giving two options for
$C_1 = \mathrm{diag}(C((t^2+1)^3), C((t^2+1)^2)) = \mathrm{diag}(C_{11}, C_{12})$
or $C_1 = \mathrm{diag}(\underbrace{C((t^2+1)^3)}_{=C_{11}}, \underbrace{C(t^2+1)}_{=C_{13}}, \underbrace{C(t^2+1)}_{=C_{13}}, \underbrace{C(t^2+1)}_{=C_{13}})$

For $f_2(t) = t^2+t+1$, $k_2 = 4$, $m_2 = 2$, partitions
are $4 = 2+2$ or $4 = 2+1+1$ giving two options for
$C_2 = \mathrm{diag}(C((t^2+t+1)^2), C((t^2+t+1)^2))$ or
$C_2 = \mathrm{diag}(\underbrace{}_{=C_{21}}\; C_{21}, C_{22} = C(t^2+t+1), C_{22})$

So get 4 options for RCF of L:

① $\mathrm{diag}(C_{11}, C_{12}, C_{21}, C_{21})$

② $\mathrm{diag}(C_{11}, C_{13}, C_{13}, C_{13}, C_{21}, C_{21})$

③ $\mathrm{diag}(C_{11}, C_{12}, C_{21}, C_{22}, C_{22})$

④ $\mathrm{diag}(C_{11}, C_{13}, C_{13}, C_{13}, C_{21}, C_{22}, C_{22})$

Inner Product Spaces and Orthogonality:

Recall the standard <u>dot product on $\mathbb{R}^n$</u>:

For $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, w = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ define $v \cdot w = \sum_{i=1}^{n} a_i b_i \in \mathbb{R}$.
$= v^T w$

<u>Properties</u>: <u>Symmetric</u>: $v \cdot w = w \cdot v$

<u>Bilinear</u>: $(c_1 v_1 + c_2 v_2) \cdot w = c_1 (v_1 \cdot w) + c_2 (v_2 \cdot w)$

and $v \cdot (c_1 w_1 + c_2 w_2) = c_1 (v \cdot w_1) + c_2 (v \cdot w_2)$

<u>Positive Definite</u>: $v \cdot v = \sum_{i=1}^{n} a_i^2 \geq 0$ and

$v \cdot v = 0$ implies $v = 0_1^n$.

<u>Def</u>. Length of $v \in \mathbb{R}^n$ is $\|v\| = \sqrt{v \cdot v}$

Distance between $v, w \in \mathbb{R}^n$ is $\|v - w\|$.

Say v is a <u>unit vector</u> when $\|v\| = 1$.

Th: (Cauchy-Schwarz Inequality)
For any $v, w \in \mathbb{R}^n$, $|v \cdot w| \leq \|v\| \cdot \|w\|$ so

$$-1 \leq \frac{v \cdot w}{\|v\| \cdot \|w\|} \leq 1 \text{ for } \|v\| \neq 0 \neq \|w\|.$$

Def: For $v, w \in \mathbb{R}^n$ the angle between v and w, $\theta_{v,w}$ is the unique angle between 0 and π such that $\cos(\theta_{v,w}) = \dfrac{v \cdot w}{\|v\| \, \|w\|}$.

Def. Say $v \perp w$ (perpendicular, orthogonal) when $\theta_{v,w} = \pi/2$, Same as $\cos(\theta_{v,w}) = 0$, iff $v \cdot w = 0$.

Def: Say $S = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is <u>orthogonal</u> when $v_i \perp v_j$ for all $1 \leq i \neq j \leq m$.

Def. Say $S=\{v_1,\dots,v_m\}\subseteq\mathbb{R}^n$ is <u>orthonormal</u> when $v_i\cdot v_j=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$
So S consists of unit vectors which are mutually perpendicular.

<u>Th</u>: If $S=\{v_1,\dots,v_m\}\subseteq\mathbb{R}^n$ is an orthogonal set of non-zero vectors then S is <u>indep.</u>

<u>Pf</u>: Suppose $\sum_{i=1}^m c_i v_i=\theta$ then for any $1\le j\le m$
$\left(\sum_{i=1}^m c_i v_i\right)\cdot v_j=\theta\cdot v_j=0$. By bilinearity, get
$\sum_{i=1}^m c_i(v_i\cdot v_j)=0$ but for $i\neq j$, $v_i\cdot v_j=0$ so
$c_j(v_j\cdot v_j)=0$. Since $v_j\neq\theta$, $v_j\cdot v_j>0$ (Pos. Definite property) so $c_j=0$, true for all $1\le j\le m$. □

Th. Suppose $S=\{v_1,\dots,v_n\}$ is an orthogonal basis of $\mathbb{R}^n$. For any $v\in\mathbb{R}^n$, $v=\sum_{i=1}^{n} c_i v_i$ and $c_j=\dfrac{v\cdot v_j}{v_j\cdot v_j}$ for each $1\le j\le n$ gives the coordinates of v with respect to S, $[v]_S$.

Pf. For each $1\le j\le n$, $v\cdot v_j=\sum_{i=1}^{n} c_i(v_i\cdot v_j)$
$=c_j(v_j\cdot v_j)$ since $v_i\cdot v_j=0$ for $i\neq j$.
Since $v_j\cdot v_j\neq 0$ (pos. def.) get $c_j=\dfrac{v\cdot v_j}{v_j\cdot v_j}$. $\square$

Cor: If $S=\{v_1,\dots,v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ then $\forall v\in\mathbb{R}^n$, $v=\sum_{i=1}^{n}(v\cdot v_i)\,v_i$.

Ex: Std. basis $S=\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is an orthonormal basis of $\mathbb{R}^n$.

Ex: For any angle ϕ let $S = \left\{ \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} \right\}$, with $v_1 = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$, $v_2 = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$

Then $v_1 \cdot v_1 = \cos^2\phi + \sin^2\phi = 1$

$v_2 \cdot v_2 = (-\sin\phi)^2 + (\cos^2\phi) = 1$

$v_1 \cdot v_2 = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$

so S is an orthonormal set in $\mathbb{R}^2$, indep, basis of $\mathbb{R}^2$. $\forall v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$

$$\begin{bmatrix} v \cdot v_1 \\ v \cdot v_2 \end{bmatrix} = \begin{bmatrix} a\cos\phi + b\sin\phi \\ -a\sin\phi + b\cos\phi \end{bmatrix} = [v]_S \text{ since}$$

$$(a\cos\phi + b\sin\phi)\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} + (-a\sin\phi + b\cos\phi)\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\phi + b\sin\phi\cos\phi + a\sin^2\phi - b\cos\phi\sin\phi \\ a\cos\phi\sin\phi + b\sin^2\phi - a\sin\phi\cos\phi + b\cos^2\phi \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Let $S = \{v_1, \dots, v_n\}$ be an orthonormal (o.n.) basis of $\mathbb{R}^n$ and let $A \in R^n_n$ be the matrix whose columns are the vectors from S, so $Col_j(A) = v_j$ for $1 \le j \le n$. Then

$$\underset{n\times 1}{Col_i(A)} \cdot \underset{n\times 1}{Col_j(A)} = \underset{1\times n}{(Col_i(A))^T}\ \underset{n\times 1}{Col_j(A)} = \delta_{ij}.$$

But $(Col_i(A))^T = Row_i(A^T)$ so

$$\underset{1\times n}{Row_i(A^T)}\ \underset{n\times 1}{Col_j(A)} = \delta_{ij}$$ is the (i,j)-entry of $\underset{(n\times n)(n\times n)}{A^T A} = [\delta_{ij}] = I_n$ = the identity matr.

so $A^T = A^{-1}$.

<u>Def.</u> We say $A \in \mathbb{R}^n_n$ is an <u>orthogonal</u> matrix when $A^T = A^{-1}$.

<u>Th</u>: $A \in \mathbb{R}^n_n$ is orthogonal iff $S = \{Col_1(A), \ldots, Col_n(A)\}$ is an <u>orthonormal</u> set in $\mathbb{R}^n$ iff $T = \{Row_1(A), \ldots, Row_n(A)\}$ is an o.n. set in $\mathbb{R}_n$ (with respect to the std dot product in $\mathbb{R}_n$).

<u>EX</u>: For each $\phi \in \mathbb{R}$, $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is orthogonal.

$$A^T A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $A^T = A^{-1}$.