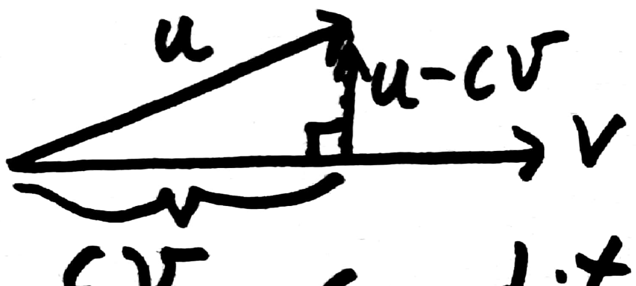
Applications of dot product to geometry:

In $\mathbb{R}^2$ can find the projection of one vector onto another as follows:



Find $c \in \mathbb{R}$ s.t. $(u - cv) \perp v$

Call $cv = \mathrm{Proj}_v(u)$.

Condition on c is that $(u - cv) \cdot v = 0$

that is, $u \cdot v - c(v \cdot v) = 0$ so $u \cdot v = c(v \cdot v)$

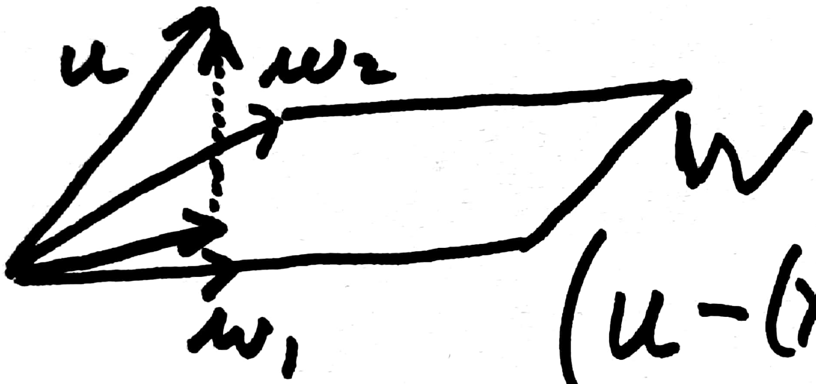
so $c = \dfrac{u \cdot v}{v \cdot v}$ if $v \cdot v \neq 0$. Can't do this if $v = 0_1^n = \theta$.

Note: If $u = xv \in \langle v \rangle$ then $\dfrac{u \cdot v}{v \cdot v} = \dfrac{(xv) \cdot v}{v \cdot v} = x$

so $\mathrm{Proj}_v(xv) = xv = u$.

In $\mathbb{R}^3$ can we find projection of $u \in \mathbb{R}^3$ onto a subspace $W = \langle w_1, w_2 \rangle$, a plane with basis $T = \{w_1, w_2\}$?

Picture:



Find $\text{Proj}_W(u)$
$= x_1 w_1 + x_2 w_2 \in W$
such that
$(u - (x_1 w_1 + x_2 w_2)) \perp W$

Equivalent conditions: $(u - x_1 w_1 - x_2 w_2) \cdot w_j = 0$ for $j = 1, 2$, iff $u \cdot w_j = x_1 (w_1 \cdot w_j) + x_2 (w_2 \cdot w_j)$ for $j = 1, 2$. This is a linear system with coefficient matrix $A = [w_i \cdot w_j]$ (symmetric)

$$\begin{bmatrix} w_1 \cdot w_1 & w_2 \cdot w_1 \\ w_1 \cdot w_2 & w_2 \cdot w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \cdot w_1 \\ u \cdot w_2 \end{bmatrix}$$

<u>Claim</u>: rank$(A)=2$ so A is invertible and this lin. sys. can be solved for any $u \in \mathbb{R}^3$.

<u>Pf</u>. If rank$(A)=1$, would have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ s.t.
$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ that means $(x_1 w_1 + x_2 w_2)\cdot w_1 = 0$
and $(x_1 w_1 + x_2 w_2)\cdot w_2 = 0$

So $(x_1 w_1 + x_2 w_2)\cdot(x_1 w_1 + x_2 w_2) =$
$x_1(x_1 w_1 + x_2 w_2)\cdot w_1 + x_2(x_1 w_1 + x_2 w_2)\cdot w_2$
$= x_1 \cdot 0 + x_2 \cdot 0 = 0$ so by pos. def. property
$x_1 w_1 + x_2 w_2 = \Theta = 0^3_1 \in \mathbb{R}^3$ (zero vector in W)
But $T = \{w_1, w_2\}$ is a basis of W, indep, so
$x_1 = 0 = x_2$, contradiction. Rank$(A)=2$ $\square$

<u>Better way</u>: If $T=\{w_1, w_2\}$ were an orthogonal basis of W, so $w_1 \cdot w_2 = 0$, then can easily solve $\begin{bmatrix} w_1\cdot w_1 & 0 \\ 0 & w_2\cdot w_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u\cdot w_1 \\ u\cdot w_2 \end{bmatrix}$ so

$$\begin{bmatrix} (w_1\cdot w_1)x_1 \\ (w_2\cdot w_2)x_2 \end{bmatrix} = \begin{bmatrix} u\cdot w_1 \\ u\cdot w_2 \end{bmatrix} \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (u\cdot w_1)/(w_1\cdot w_1) \\ (u\cdot w_2)/(w_2\cdot w_2) \end{bmatrix}$$

$$\mathrm{Proj}_W(u) = \left(\frac{u\cdot w_1}{w_1\cdot w_1}\right) w_1 + \left(\frac{u\cdot w_2}{w_2\cdot w_2}\right) w_2$$

$$= \mathrm{Proj}_{w_1}(u) + \mathrm{Proj}_{w_2}(u)$$

This projection problem can be solved for any subspace $W \leq \mathbb{R}^n$.

Th: Let $T = \{w_1, \dots, w_m\}$ be a basis of subspace W in $\mathbb{R}^n$. For any $u \in \mathbb{R}^n$ can find $\mathrm{Proj}_W(u) = \sum_{i=1}^{m} x_i w_i \in W$ such that $\left(u - \sum_{i=1}^{m} x_i w_i\right) \perp W$, that is, for each $1 \le j \le m$,

$$\sum_{i=1}^{m} x_i (w_i \cdot w_j) = u \cdot w_j .$$

This is the lin. sys. $AX = B$ where $A = [w_i \cdot w_j] \in \mathbb{R}^m_m$, (is invertible) $X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $B = \begin{bmatrix} u \cdot w_1 \\ \vdots \\ u \cdot w_m \end{bmatrix}$. If T is orthogonal basis then A is diagonal and $x_i = \dfrac{u \cdot w_i}{w_i \cdot w_i}$

so $$\mathrm{Proj}_W(u) = \sum_{i=1}^{m} \left(\frac{u \cdot w_i}{w_i \cdot w_i}\right) w_i .$$

Example: In $\mathbb{R}^3$ let $W = \{X \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

$= v^\perp$ for $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Let $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

Find $\mathrm{Proj}_W(u)$.

Step ①: Pick a basis for W. Solve $[1\ 1\ 1 \mid 0]$

$$\left.\begin{array}{l} x_1 = -r-s \\ x_2 = r \in \mathbb{R} \\ x_3 = s \in \mathbb{R} \end{array}\right\} \text{ so } W = \left\{ \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; r, s \in \mathbb{R} \right\} = \Big\langle \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{w_2} \Big\rangle$$

$\mathrm{Proj}_W(u) = x_1 w_1 + x_2 w_2$ with x_1 and x_2 determined by conditions: $\big(u - (x_1 w_1 + x_2 w_2)\big) \cdot w_j = 0$

$\Leftrightarrow x_1 (w_1 \cdot w_j) + x_2 (w_2 \cdot w_j) = u \cdot w_j$ for $j = 1, 2$.

$$\Leftrightarrow \begin{bmatrix} w_1 \cdot w_1 & w_2 \cdot w_1 \\ w_1 \cdot w_2 & w_2 \cdot w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \cdot w_1 \\ u \cdot w_2 \end{bmatrix}$$

$A = [w_i \cdot w_j] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} u \cdot w_1 \\ u \cdot w_2 \end{bmatrix} = \begin{bmatrix} -a+b \\ -a+c \end{bmatrix}$ so

solve $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b-a \\ c-a \end{bmatrix}$. Either use A^{-1}:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} b-a \\ c-a \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b-a \\ c-a \end{bmatrix}$$ or

row reduce $\left[\begin{array}{cc|c} 2 & 1 & b-a \\ 1 & 2 & c-a \end{array}\right]$ to $\left[\begin{array}{cc|c} 1 & 0 & (-a+2b-c)/3 \\ 0 & 1 & (-a-b+2c)/3 \end{array}\right]$

$$\mathrm{Proj}_W(u) = \frac{1}{3}(-a+2b-c)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}(-a-b+2c)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

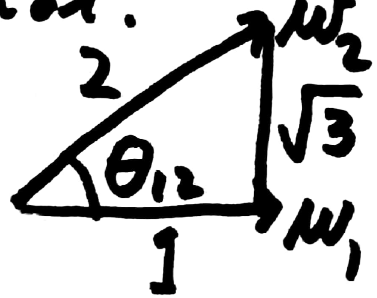
$$= \frac{1}{3}\begin{bmatrix} 2a-b-c \\ -a+2b-c \\ -a-b+2c \end{bmatrix} \in W.$$

Note: $\mathrm{Proj}_W\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathrm{Proj}_W(u) = u$ when $u \in W$.

Can we find an orthogonal basis of W ?

$$T = \left\{ w_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, w_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} \right\}$$ but $w_1 \cdot w_2 = 1$ so not orthogonal.

$$\cos\left(\theta_{w_1, w_2}\right) = \frac{w_1 \cdot w_2}{\|w_1\|\,\|w_2\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$



But $\left(w_2 - \mathrm{Proj}_{w_1}(w_2)\right) \perp w_1$

Let $$w_2' = w_2 - \left(\frac{w_2 \cdot w_1}{w_1 \cdot w_1}\right) w_1 = w_2 - \frac{1}{2} w_1 = \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$

Then $$T' = \left\{ w_1' = w_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, w_2' = \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} \right\}$$ is an orthogonal basis of W.

$\mathrm{Proj}_W(u) = x_1 w_1' + x_2 w_2'$ is solved easily:

$$x_1 = (u \cdot w_1')/(w_1' \cdot w_1') = (b-a)/2$$

$$x_2 = (u \cdot w_2')/(w_2' \cdot w_2') = \left(c - \frac{a}{2} - \frac{b}{2}\right)/\left(3/2\right)$$

$$\mathrm{Proj}_W(u) = \frac{(b-a)}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} + \frac{2}{3}\left(\frac{2c-a-b}{2}\right)\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$

$$= \frac{(3b-3a)}{6}\begin{bmatrix}-1\\1\\0\end{bmatrix} + \frac{(2c-a-b)}{6}\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3a-3b+a+b-2c\\-3a+3b+a+b-2c\\-2a-2b+4c\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}2a-b-c\\-a+2b-c\\-a-b+2c\end{bmatrix}$$ is same answer as before.

Th. Let $W \leq \mathbb{R}^n$ have basis $T = \{w_1, \dots, w_m\}$.
Then an orthogonal basis $T' = \{w_1', \dots, w_m'\}$
for W can be found as follows:

$$w_1' = w_1 \quad , \quad w_2' = w_2 - \mathrm{Proj}_{w_1'}(w_2) = w_2 - \left(\frac{w_2 \cdot w_1'}{w_1' \cdot w_1'}\right) w_1'$$

$$w_3' = w_3 - \mathrm{Proj}_{\langle w_1', w_2' \rangle}(w_3) =$$

$$w_3 - \left(\frac{w_3 \cdot w_1'}{w_1' \cdot w_1'}\right) w_1' - \left(\frac{w_3 \cdot w_2'}{w_2' \cdot w_2'}\right) w_2'$$

$\vdots$

$$w_i' = w_i - \mathrm{Proj}_{\langle w_1', \dots, w_{i-1}' \rangle}(w_i) = w_i - \sum_{j=1}^{i-1} \left(\frac{w_i \cdot w_j'}{w_j' \cdot w_j'}\right) w_j'$$

for $1 \leq i \leq m$. Also,

$\langle w_1, \dots, w_i \rangle = \langle w_1', \dots, w_i' \rangle$ for $1 \leq i \leq m$.

Pf. We have defined $\mathrm{Proj}_{w_1'}(w_2)$ such that
$(w_2 - \mathrm{Proj}_{w_1'}(w_2)) \perp w_1'$ so $w_2' \perp w_1'$ makes $\{w_1', w_2'\}$ an orthogonal basis for its span. But then $(w_3 - \mathrm{Proj}_{\langle w_1', w_2' \rangle}(w_3)) \perp \langle w_1', w_2' \rangle$ so $\{w_1', w_2', w_3'\}$ is an orthogonal basis for its span. The formula given for that projection was proved before based on the assumption that basis $\{w_1', w_2', w_3'\}$ is orthogonal. The formula for w_i' follows by the same argument (by induction) since $\{w_1', \dots, w_{i-1}'\}$ is an orthogonal basis of its span. That formula also shows that $w_i' \in \langle w_1', \dots, w_{i-1}', w_i \rangle = \langle w_1, \dots, w_{i-1}, w_i \rangle$ and $w_i \in \langle w_1', \dots, w_{i-1}', w_i' \rangle$ so

$$\langle w_1, \dots, w_i \rangle = \langle w_1', \dots, w_i' \rangle. \quad \square$$

Example: Let $W \leq \mathbb{R}^4$ where a basis of W is

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

Then G.-S. process is:

$$w_1' = w_1, \quad w_2' = w_2 - \left(\frac{w_2 \cdot w_1'}{w_1' \cdot w_1'}\right) w_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{2}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$w_3' = w_3 - \left(\frac{w_3 \cdot w_1'}{w_1' \cdot w_1'}\right) w_1' - \left(\frac{w_3 \cdot w_2'}{w_2' \cdot w_2'}\right) w_2'$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \left(\frac{4}{2}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{6}{2}\right) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

so $T' = \left\{ w_1' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, w_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, w_3' = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis of W. Check that $w_i' \cdot w_j' = 0$ for $i \neq j$.

Th. (Normalization Step of G.-S.)
After obtaining orthogonal basis T' from the G.-S. process, we can replace each vector $w_i' \in T'$ by a unit vector, $w_i'' = \frac{w_i'}{\|w_i'\|}$, to get an orthonormal basis $T'' = \{w_1'', \dots, w_m''\}$ of W.

EX: In the last example, $\|w_1'\| = \sqrt{2} = \|w_2'\|$ and $\|w_3'\| = \sqrt{4} = 2$ so an orthonormal basis of W is

$$T'' = \left\{ w_1'' = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\ w_2'' = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\ w_3'' = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} \right\}$$

Exercise: Use T' to find $x_1, x_2, x_3 \in \mathbb{R}$ s.t. $\mathrm{Proj}_W(v) = x_1 w_1' + x_2 w_2' + x_3 w_3'$ for any $v = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \in \mathbb{R}^4$.