Example: For $W = \langle T \rangle = \langle T' \rangle = \langle T'' \rangle$ as before, find $W^\perp = \{X \in \mathbb{R}^4 \mid X \perp W\}$ and use its basis vector to extend T' to an orthogonal basis of $\mathbb{R}^4$, extend T'' to an orthonormal basis of $\mathbb{R}^4$, and use that answer to give an orthogonal matrix $A \in \mathbb{R}^4_4$.

Solution: Find $W^\perp = \{X \in \mathbb{R}^4 \mid X \cdot w_i' = 0, 1 \le i \le 3\}$ by solving

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = r \\ x_2 = -r \\ x_3 = -r \\ x_4 = r \in \mathbb{R} \end{array}$$

$$W^\perp = \left\{ \begin{bmatrix} r \\ -r \\ -r \\ r \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; r \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle \text{ so}$$

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$ is an orthogonal basis of $\mathbb{R}^4$ extending T'.

ormalizing each vector in S' gives o.n. basis

$$S'' = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \right\}$$ of $\mathbb{R}^4$, and

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/2 & -1/2 \\ 1/\sqrt{2} & 0 & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

is an orthogonal matrix in $\mathbb{R}^4$ whose columns are the vectors in o.n. set S''.

Check that $AA^T = I_4 = A^TA$

Important results about orthog. matrices:

Th: For any $X, Y \in \mathbb{R}^n$ and for any $A \in \mathbb{R}^n_n$, we have $(AX)\cdot Y = X \cdot (A^T Y)$.

Pf. $(AX)\cdot Y = (AX)^T Y = (X^T A^T) Y = X^T(A^T Y) = X\cdot(A^T Y)$

Th: For any $X, Y \in \mathbb{R}^n$ if $A \in \mathbb{R}^n_n$ is orthogonal then $(AX)\cdot(AY) = X\cdot Y$.

Pf. Since A orthog. means $A^T = A^{-1}$, we have $(AX)\cdot(AY) = X\cdot(A^T A Y) = X\cdot(A^{-1}AY) = X\cdot(I_n Y) = X\cdot Y$.

Cor: If $A^T = A^{-1}$ then for any $X, Y \in \mathbb{R}^n$ we have $\|AX\| = \|X\|$ and $\theta_{X,Y} = \theta_{AX,AY}$ so $L_A : \mathbb{R}^n \to \mathbb{R}^n$ preserves lengths and angles.

Important result about symm. matrices: $\boxed{300}$

<u>Th</u>: Let $A = A^T \in \mathbb{R}^n_n$ be symmetric and let $\lambda \neq \mu$ in $\mathbb{R}$ be e-values of A. Then the e-spaces A_λ and A_μ are perpendicular, $A_\lambda \perp A_\mu$.

<u>Pf</u>. Need to show that for any $X \in A_\lambda$, $Y \in A_\mu$ that $X \cdot Y = 0$. It is clear if $X = 0^n_1$ or $Y = 0^n_1$ so suppose $AX = \lambda X$ and $AY = \mu Y$ for $X, Y \in \mathbb{R}^n$ nonzero e-vectors. Then we have

$$\lambda(X \cdot Y) = (\lambda X) \cdot Y = (AX) \cdot Y = X \cdot (A^T Y) = X \cdot (AY) = X \cdot (\mu Y)$$
$$= \mu(X \cdot Y)$$

So $\lambda(X \cdot Y) = \mu(X \cdot Y)$ so

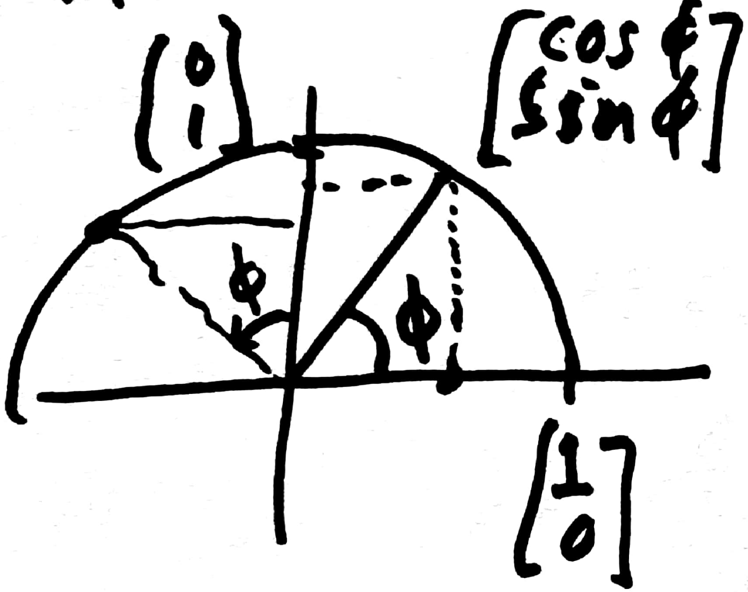
$(\lambda - \mu)(X \cdot Y) = 0$. But $\lambda - \mu \neq 0$ so $\overline{X \cdot Y = 0}$ $\square$

Th: Let $A = A^T \in \mathbb{R}^n_n$ and suppose $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ are the distinct e-values of A. Let T_i' be an orthonormal basis of e-space A_{λ_i}, $1 \leq i \leq r$, obtained by Gram-Schmidt process from any basis T_i of A_{λ_i}. Then $T' = T_1' \cup T_2' \cup \dots \cup T_r'$ is an orthonormal basis of $\mathbb{R}^n$ and $P = {}_S P_{T'}$ is an orthogonal matrix (whose columns are the vectors in T') such that $P^{-1} A P = D$ is diagonal with blocks $\lambda_i I_{g_i}$ on the diagonal, $g_i = \dim(A_{\lambda_i}) = k_i$ (geom. = alg. mult.).

Since P is orthog. $P^{-1} = P^T$ so $D = P^T A P$ and we say A can be "orthogonally diag-ized."

Pf. In Advanced Lin. Alg. it is shown that all e-values of symm. $A \in \mathbb{R}^n_n$ are real, and that $g_i = k_i$ so A is diag-able. Since G.-S. gives orthonormal bases T_i' for each A_{λ_i}, and $A_{\lambda_i} \perp A_{\lambda_j}$ for $1 \le i \ne j \le r$ by last Theorem, we get that T' is an orthonormal set of n e-vectors in $\mathbb{R}^n$ so $P = {}_S P_{T'}$ is orthog., $P^{-1} = P^T$ and $D = P^T A P$ is diag. with the e-values λ_i on the diag. repeated $g_i = k_i$ times in blocks corresponding to the order of e-vectors in T'. $\square$

$L_A : \mathbb{R}^2 \to \mathbb{R}^2$ for $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$



$$L_A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathrm{Col}_1(A) = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

$$L_A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{Col}_2(A) = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\phi + \pi/2) \\ \sin(\phi + \pi/2) \end{bmatrix}$$

$\{L_A(e_1), L_A(e_2)\}$ is another o.n. basis of $\mathbb{R}^2$, just $S = \{e_1, e_2\}$ rotated (c.c.w) by angle ϕ. This L_A preserves lengths and angles

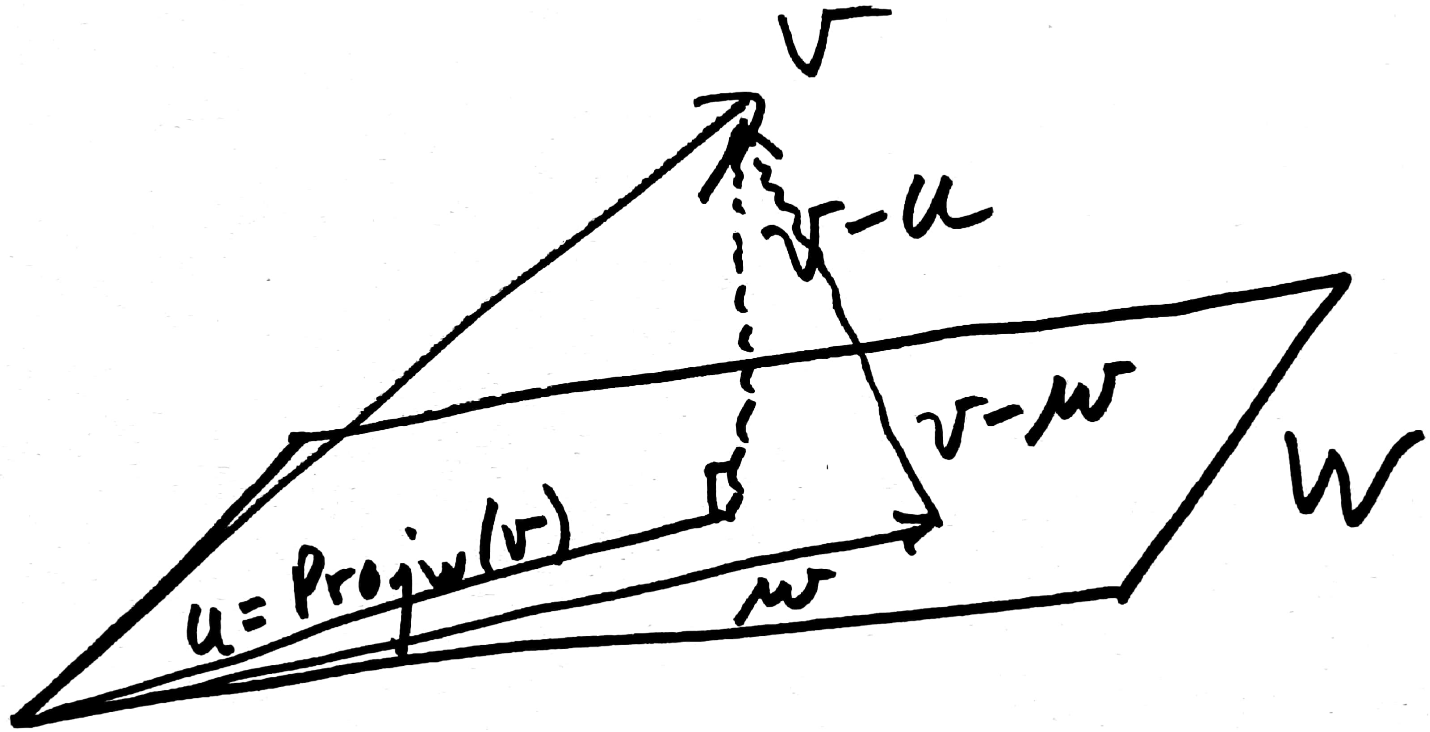
Ex: Reflections: $L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is refl. w.r.t. $y = x$, so $L_A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has columns $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ an. o.n. basis of $\mathbb{R}^2$

$A^T = A^{-1}$ so A is orthog. matrix.

L_A preserved lengths & angles.

Meaning of $Proj_W(v)$ as "best approximation to v in W"



$$\|v-u\| < \|v-w\| \quad \text{if } w \neq u \quad \forall w \in W$$

$\leq$

"Best approx. Thm"

Th (Pythagorean Thm in $\mathbb{R}^n$).
For $X, Y \in \mathbb{R}^n$, if $X \cdot Y = 0$ (so $X \perp Y$) then $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$. For $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$, if $\{v_1, \ldots, v_m\}$ is orthogonal (so $v_i \cdot v_j = 0$ for $i \neq j$) then $\|\sum_{i=1}^m v_i\|^2 = \sum_{i=1}^m \|v_i\|^2$.

Pf. $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + X\cdot Y + Y\cdot X + Y\cdot Y$
$= X\cdot X + Y\cdot Y = \|X\|^2 + \|Y\|^2$. The general case of m orthogonal vectors follows by induction, using $X = v_1 + \cdots + v_{m-1}$, $Y = v_m$. $\square$

Th (Triangle Inequality in $\mathbb{R}^n$)

For any $X, Y \in \mathbb{R}^n$, $\|X+Y\| \le \|X\| + \|Y\|$.

Pf. $\|X+Y\|^2 = \|X\|^2 + 2(X \cdot Y) + \|Y\|^2$

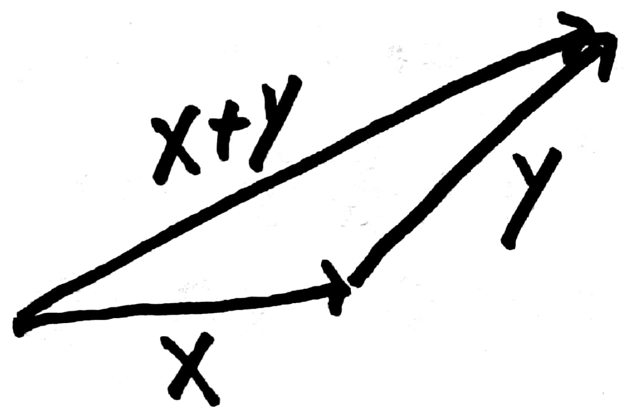
$\le \|X\|^2 + 2|X \cdot Y| + \|Y\|^2$ so by

Cauch-Schw. Ineq. $\le \|X\|^2 + 2(\|X\|)(\|Y\|) + \|Y\|^2$

$= (\|X\| + \|Y\|)^2$. This gives

$0 \le \|X+Y\| \le \|X\| + \|Y\|$. $\square$

Geometrical Picture:



Complex Numbers: $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R},\ i^2=-1\}$ 308

The "imaginary" number $i \in \mathbb{C}$, $i \notin \mathbb{R}$, is a special number such that $i^2 = -1$.

$\mathbb{C}$ is a "field", like $\mathbb{R}$ = real numbers and $\mathbb{Q}$ = rational numbers, where we can do arithmetic, Use for scalars in Lin. Alg.

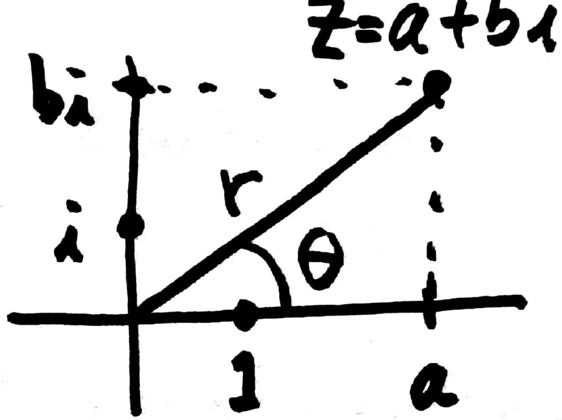
Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$

Mult: $(a+bi)\cdot(c+di) = ac + adi + bci + bdi^2$
(commutative) $= (ac-bd)+(ad+bc)i$

<u>Def.</u> For $z = a+bi$ let "complex conjugate" of z be $\overline{z} = a-bi$, so $z\overline{z} = a^2+b^2 \geq 0$ and $z\overline{z} = 0$ iff $z = 0+0i = 0$.

Note: For $0 \neq z = a+bi \in \mathbb{C}$, $z\bar{z} > 0$ and $z^{-1} = \dfrac{\bar{z}}{a^2+b^2} \in \mathbb{C}$ is mult. inverse of z.

Ex: If $z = 3+4i$ then $z\bar{z} = 3^2+4^2 = 25$ so $z\left(\dfrac{\bar{z}}{25}\right) = 1$, $z^{-1} = \dfrac{3-4i}{25}$.



Graphical Picture of $\mathbb{C}$:

looks like $\mathbb{R}^2$ with $1 \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix}$

$i \leftrightarrow \begin{bmatrix}0\\1\end{bmatrix}$

but $\mathbb{C}$ has a <u>mult</u> while $\mathbb{R}^2$ does <u>not</u>.

Related to "polar coordinates"

$z = \underbrace{(r\cos\theta)}_{=\,a} + \underbrace{(r\sin\theta)}_{b}\, i$

$= a + b\,i$

Complex vector spaces: <u>Definition</u>

Say $(V, +, \cdot, \theta)$ is a <u>complex</u> vector space (or V is a vector space over $\mathbb{C}$) when V obeys all the usual vector space axioms where scalars are in $\mathbb{C}$ (instead of in $\mathbb{R}$).

<u>Ex 1</u>: $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \middle| \; z_j \in \mathbb{C}, 1 \le j \le n \right\}$ with the usual $+$ and $\cdot$.

<u>Ex 2</u>: $\mathbb{C}^m_n = \{ A = [a_{ij}] \mid a_{ij} \in \mathbb{C}, 1 \le i \le m, 1 \le j \le n \}$

$=$ $m \times n$ complex matrices. As before,

For $A \in \mathbb{C}^m_n$, $L_A : \mathbb{C}^n \to \mathbb{C}^m$ is $L_A(X) = AX$.

Can do <u>any</u> linear algebra problem for complex vector spaces:

Solve linear system $AX = B$,

Find $\mathrm{Ker}(L)$, $\mathrm{Range}(L)$ for any linear map $L : V \to W$ for complex v. spaces V and W,

Find a basis for a subspace $U \leq V$,

For bases S in V, T in W, $L : V \to W$, find ${}_T[L]_S \in \mathbb{C}^m_n$ if $\dim(V) = n$, $\dim(W) = m$.

Have <u>std</u> basis $S = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{C}^n$

since $\mathbb{C}^n = \left\{ Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sum_{j=1}^{n} z_j e_j \;\middle|\; z_j \in \mathbb{C},\ 1 \leq j \leq m \right\}$.

Ex: For $A = \begin{bmatrix} i & 1+i & 1-i \\ 1+i & 1-i & 1 \end{bmatrix}$ Solve $AX = 0$; $X \in \mathbb{C}^3$

$$\left[\begin{array}{ccc|c} i & 1+i & 1-i & 0 \\ 1+i & 1-i & 1 & 0 \end{array}\right] \quad \begin{array}{l} -iR_1 \to R_1 \\ (1-i)R_2 \to R_2 \end{array}$$

using $(1-i)(1-i) = 1-1-2i = -2i$

$$\to \left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 2 & -2i & 1-i & 0 \end{array}\right] \quad -2R_1 + R_2 \to R_2$$

$$\begin{array}{ccc} -2 & -2+2i & 2+2i \end{array} \quad + \quad \begin{array}{ccc} 0 & -1+i & 2-i \end{array}$$

$$\boxed{-\tfrac{1}{2}(3+i)(-1+i) = -\tfrac{1}{2}(-4+2i) = 2-i}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 0 & -2 & 3+i & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1-i & -1-i & 0 \\ 0 & 1 & -\tfrac{1}{2}(3+i) & 0 \end{array}\right] \quad (-1+i)R_3 + R_1 \to R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1-2i & 0 \\ 0 & 1 & -\tfrac{1}{2}(3+i) & 0 \end{array}\right] \quad \begin{array}{l} x_1 = (-1+2i)z \\ x_2 = \tfrac{1}{2}(3+i)z \\ x_3 = z \in \mathbb{C} \text{ free} \end{array}$$

$\dim(Nul(A)) = 1$

$$Nul(A) = \left\{ z \begin{bmatrix} -1+2i \\ \tfrac{1}{2}(3+i) \\ 1 \end{bmatrix} \in \mathbb{C}^3 \;\middle|\; z \in \mathbb{C} \right\} = \left\langle \begin{bmatrix} -1+2i \\ \tfrac{3}{2} + \tfrac{1}{2}i \\ 1 \end{bmatrix} \right\rangle$$

Std. dot product in $\mathbb{C}^n$:

For $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, $W = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ define

$$Z \cdot W = \sum_{j=1}^{n} z_j \overline{w_j}$$ (note complex conjugate on W coordinates)

$= Z^T \overline{W}$ where $\overline{W} = \begin{bmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{bmatrix}$. $\forall a, b \in \mathbb{C}$,

<u>Then</u>: $(aZ + bZ') \cdot W = a(Z \cdot W) + b(Z' \cdot W)$ but

$Z \cdot (aW + bW') = \bar{a}(Z \cdot W) + \bar{b}(Z \cdot W')$

called "sesquilinear", linear in first input, conjugate linear in second input.

$$\cdot : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

Also; $Z \cdot W = \overline{W \cdot Z}$ (conjugate symm.)

and $Z \cdot Z = \sum_{j=1}^{n} z_j \overline{z}_j = \sum_{j=1}^{n} (a_j^2 + b_j^2) \geq 0$ (real)

where $z_j = a_j + b_j i$, and $Z \cdot Z = 0$ iff $Z = 0_1^n$

called "positive definite".

This dot product gives geometry on $\mathbb{C}^n$:

$\|Z\| = \sqrt{Z \cdot Z} \geq 0$ length, etc.

Important advantage working over $\mathbb{C}$ is all polynomials factor into linear factors.

Ex: $X^2 + 1 = (X + i)(X - i)$

$ax^2+bx+c=0$ has roots $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

in $\mathbb{R}$ when $b^2-4ac \geq 0$

in $\mathbb{C}$ when $b^2-4ac<0$.

Application to diagonalization:

__Ex.__ $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ has $\mathrm{Char}_A(\lambda)=\begin{vmatrix}\lambda & -1\\ 1 & \lambda\end{vmatrix}=\lambda^2+1$

has two distinct complex $=(\lambda+i)(\lambda-i)$

e-values, $\lambda_1=-i$, $\lambda_2=i$.

Espaces: A_{λ_1}: $\left[\begin{array}{cc|c}-i & -1 & 0\\ 1 & -i & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right]$ $x_1=iz$, $x_2=z\in\mathbb{C}$ free

$A_{\lambda_1}=\left\langle\begin{bmatrix}i\\ 1\end{bmatrix}\right\rangle$. A_{λ_2}: $\left[\begin{array}{cc|c}i & -1 & 0\\ 1 & i & 0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1 & i & 0\\ 0 & 0 & 0\end{array}\right]$ $x_1=-iz$, $x_2=z\in\mathbb{C}$ free

$A_{\lambda_2}=\left\langle\begin{bmatrix}-i\\ 1\end{bmatrix}\right\rangle$ Get e-basis $T=\left\{\begin{bmatrix}i\\ 1\end{bmatrix},\begin{bmatrix}-i\\ 1\end{bmatrix}\right\}$ for $\mathbb{C}^2$

If $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is std basis of $\mathbb{C}^2$,

transition matrix $P = {}_S P_T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

has inverse $P^{-1} = {}_T P_S = \frac{1}{2i} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$

and $P^{-1} A P = \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D$ is diagonal.

So working over $\mathbb{C}$ allows more matrices to be diagonalizable, but still not all.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \text{char}_A(\lambda) = \begin{vmatrix} (\lambda-1) & -1 \\ 0 & (\lambda-1) \end{vmatrix} = (\lambda-1)^2$

has only e-value $\lambda_1 = 1$, $k_1 = 2$

A_{λ_1}: $\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x_1 = z \in \mathbb{C} \text{ free} \\ x_2 = 0 \end{array}$ $\quad A_{\lambda_1} = \left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbb{C} \right\}$

$g_1 = 1 < 2 = k_1$ $\quad = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$

Cannot find a basis of $\mathbb{C}^2$
consisting of e-vectors for A.
A is not diag-able over $\mathbb{C}$.