Th. In $\mathbb{C}^n$ with its standard dot product, 318

$z \cdot w = z^T \overline{w}$, for any $A \in \mathbb{C}^n_n$ we have

$(Az) \cdot w = z \cdot (\overline{A}^T w)$.

Pf. $(Az) \cdot w = (Az)^T \overline{w} = (z^T A^T)\overline{w} = z^T (A^T \overline{w})$

$= z^T \left(\overline{\overline{A}^T w}\right) = z \cdot (\overline{A}^T w)$. $\square$

Def. For $A \in \mathbb{C}^n_n$, define the __Hermitian conjugate__, $A^H = \overline{A}^T = A^*$.

Def. Say $A \in \mathbb{C}^n_n$ is __Hermetian__ when $A^H = A$, __skew-Hermetian__ (or anti-Hermetian) when $A^H = -A$, __unitary__ when $A^H = A^{-1}$.

Th. If $A \in \mathbb{C}^n_n$ is unitary then $\forall Z, W \in \mathbb{C}^n$, $(AZ)\cdot(AW) = Z \cdot W$.

Pf. $(AZ)\cdot(AW) = Z \cdot (A^H A W) = Z \cdot (A^{-1} A W) = Z \cdot W$. □

Th. $A \in \mathbb{C}^n_n$ is unitary iff $\{Col_j(A) \mid 1 \le j \le n\}$ is an orthonormal set in $\mathbb{C}^n$ w.r.t. the std. dot product.

Pf. $Col_i(A) \cdot Col_j(A) = Col_i(A)^T \overline{Col_j(A)}$
$= Row_i(A^T)\, Col_j(\bar{A}) = \delta_{ij}$ iff $A^T \bar{A} = I_n$
iff $\bar{A}^T A = I_n$ iff $A^H = A^{-1}$. □

Th. If $A = A^T \in \mathbb{R}^n_n$ then all eigenvalues of A are real.

Pf. Considering $A \in \mathbb{C}^n_n$, $\mathrm{Char}_A(t) = \prod_{i=1}^{r} (t-\lambda_i)^{\mu_i}$ for $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ the distinct e-values of A. Let $\theta \neq X \in \mathbb{C}^n$ be an e-vector for A with e-value $\lambda_i \in \mathbb{C}$, so $AX = \lambda_i X$. Then

$$\lambda_i (X \cdot X) = (\lambda_i X) \cdot X = (AX) \cdot X = X \cdot (A^H X) = X \cdot (AX)$$

$$= X \cdot (\lambda_i X) = \overline{\lambda_i} (X \cdot X) \text{ since } A^H = \bar{A}^T = A^T = A,$$

so $(\lambda_i - \overline{\lambda_i})(X \cdot X) = 0$. But $X \cdot X > 0$ so $\lambda_i = \overline{\lambda_i}$ means $\lambda_i \in \mathbb{R}$. ∎

Def. Let $S \subseteq \mathbb{R}^n$. Define "S perp" to be
$S^\perp = \{X \in \mathbb{R}^n \mid X \cdot Y = 0, \forall Y \in S\}$.

Th. For any nonempty $S \subseteq \mathbb{R}^n$, $S^\perp = \langle S \rangle^\perp \leq \mathbb{R}^n$.

Pf. $\theta = 0_1^n \in S^\perp$ since $0_1^n \cdot Y = 0$, $\forall Y \in S$.
$\forall X_1, X_2 \in S^\perp$, $(X_1 + X_2) \cdot Y = (X_1 \cdot Y) + (X_2 \cdot Y) = 0 + 0 = 0$
$\forall Y \in S$ so $X_1 + X_2 \in S^\perp$ (closed under $+$).
$\forall X \in S^\perp$, $\forall a \in \mathbb{R}$, $(aX) \cdot Y = a(X \cdot Y) = a(0) = 0$
$\forall Y \in S$ so $aX \in S^\perp$ (closed under scalar prod. $\cdot$)

$\forall Y_1, \ldots, Y_m \in S$, $\forall a_1, \ldots, a_m \in \mathbb{R}$, $\forall X \in S^\perp$, $X \cdot Y_i = 0$
for $1 \leq i \leq m$, so $X \cdot \sum_{i=1}^m a_i Y_i = \sum_{i=1}^m a_i (X \cdot Y_i) = 0$ so $X \in \langle S \rangle^\perp$,
so $S^\perp \leq \langle S \rangle^\perp$.

Also, $S \subseteq \langle S \rangle$, so $X \in \langle S \rangle^{\perp}$ implies $X \in S^{\perp}$

so $\langle S \rangle^{\perp} \subseteq S^{\perp}$ giving $S^{\perp} = \langle S \rangle^{\perp}$. $\square$

<u>Note</u>. If $S = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$ then $X \in S^{\perp}$ iff $v_i \cdot X = 0$ for $1 \le i \le m$ iff $v_i^T X = 0$, $1 \le i \le m$ is the homog. lin. sys. $AX = 0$ where $Row_i(A) = v_i^T$, $A \in \mathbb{R}^m_n$.

<u>Ex</u>: If $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ then find $S^{\perp}$ by solving

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = r \\ x_2 = -2r \\ x_3 = r \in \mathbb{R} \end{array}$$

so $S^{\perp} = \left\{ \begin{bmatrix} r \\ -2r \\ r \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle$

and $\mathbb{R}^3 = \langle S \rangle \oplus S^{\perp}$.

<u>Def</u>. Let $W_1, \dots, W_t \le \mathbb{R}^n$, and $W = W_1 \oplus \cdots \oplus W_t \le \mathbb{R}^n$. If $W_i \perp W_j$ for $1 \le i \ne j \le t$ then say $W = W_1 \oplus \cdots \oplus W_t$ is an <u>orthogonal direct sum</u>.

<u>Note</u>: If $W_1 \perp W_2$ and $W_1 \perp W_3$ then $W_1 \perp (W_2 + W_3)$ and similarly for larger sums. So for mutually orthog. subspaces any sum is a direct sum.

<u>Th</u>: For any subspace $W \le \mathbb{R}^n$, $W + W^\perp = \mathbb{R}^n$ is an orthog. direct sum decomposition of $\mathbb{R}^n$.

<u>Pf</u>. This is clear if $W = \{\theta\}$ is trivial.
Suppose $\{\theta\} \lneq W \lneq \mathbb{R}^n$.

$\forall X \in \mathbb{R}^n$, $Proj_W(X) \in W$ is defined so that $X - Proj_W(X) \in W^\perp$ so

$X = Proj_W(X) + (X - Proj_W(X)) \in W + W^\perp$ shows $\mathbb{R}^n = W + W^\perp$. Clearly, $W \perp W^\perp$ by definition and $v \in W \cap W^\perp$ implies $v \cdot v = 0$ so $v = \theta$ so $W \cap W^\perp = \{\theta\}$. $\mathbb{R}^n = W \oplus W^\perp$ is an orthog. direct sum. $\square$

Let's try to apply these concepts to the eigenspaces $A_{\lambda_1}, \dots, A_{\lambda_r}$ for $A = A^T \in \mathbb{R}^n_n$ where we know $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ and $Char_A(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$.

$\mathbb{R}^n = A_{\lambda_1} \oplus A_{\lambda_1}^\perp$ is an orthog. direct sum.

<u>Claim</u>: $A_{\lambda_1}^{\perp}$ is an A-invariant subspace,
that is, $\forall Y \in A_{\lambda_1}^{\perp}$, $AY \in A_{\lambda_1}^{\perp}$.

<u>Pf.</u> Let $Y \in A_{\lambda_1}^{\perp}$ and $X \in A_{\lambda_1}$. Then
$X \cdot (AY) = (A^T X) \cdot Y = (AX) \cdot Y = (\lambda_1 X) \cdot Y = \lambda_1 (X \cdot Y) = 0.$

Let $T_1 = \{v_{11}, \dots, v_{1g_1}\}$ be one basis of A_{λ_1} and $T_1^{\perp}$ any basis of $A_{\lambda_1}^{\perp}$ so $T_1 \cup T_1^{\perp} = S_1$ is a basis of $\mathbb{R}^n$ s.t. $L = L_A : \mathbb{R}^n \to \mathbb{R}^n$ has ${}_{S_1}[L]_{S_1}$ in block diag. form $\left[\begin{array}{c|c} \lambda_1 I_{g_1} & 0 \\ \hline 0 & B \end{array}\right]$ with $B = {}_{T_1^{\perp}}\left[L|_{A_{\lambda_1}^{\perp}}\right]_{T_1^{\perp}}$.

Also, $Char_A(t) = Char_{\lambda_1 I_{g_1}}(t) \cdot Char_B(t) = (t-\lambda_1)^{g_1} Char_B(t).$

If $g_1 < k_1$ then $(t-\lambda_1)$ divides $Char_B(t)$ so $A_{\lambda_1}^{\perp}$ would have to contain a λ_1-e-vector for A, contradicting $A_{\lambda_1} \cap A_{\lambda_1}^{\perp} = \{\theta\}$. So $g_1 = k_1$ and we are reduced to studing the restriction of L_A to $A_{\lambda_1}^{\perp}$ where the $Char_B(t) = \prod_{i=2}^{r}(t-\lambda_i)^{k_i}$.

The problem is that we cannot just replace A by B because B need not be symmetric.

Use Gram-Schmidt process to change T_1 and $T_1^{\perp}$ into orthonormal bases of A_{λ_1} and $A_{\lambda_1}^{\perp}$ respectively. Then S_1 would be an o.n. basis of $\mathbb{R}^n$ and the transition matrix ${}_S P_{S_1} = P$ would be orthogonal, so $P^{-1} = P^T$. It means the block diag. form similar to A is P^TAP so $(P^TAP)^T = P^TA^TP = P^TAP$ says it is symmetric. Then $B = B^T$ in the corner, and we can apply these arguments to B. Inductively, we get B is orthogonally diag-able, each $g_i = k_i$. ∎

Def. Say $A \in \mathbb{C}^n_n$ is *normal* when $AA^H = A^HA$, that is, A commutes with $\bar{A}^T$.

Note. A unitary implies A is normal, but not conversely.

See Section 13.9 of our textbook for more results about when $A \in \mathbb{C}^n_n$ is guaranteed to be diag-able.