General Inner Product Spaces over $\mathbb{R}$:

Let V be a real vectorspace.

Def. Say V is a (real) inner product space (IPS) when there is a bilinear function $(\cdot,\cdot): V \times V \to \mathbb{R}$ such that $(v_1, v_2) = (v_2, v_1)$ $\forall v_1, v_2 \in V$ (symmetric) and $(v, v) \geq 0$, $\forall v \in V$, with $(v, v) = 0$ iff $v = \theta$ (positive definite).

In that case, $(\cdot,\cdot)$ is called the inner product on V, and can be used to define

$\|v\| = \sqrt{(v,v)}$ the length of $v \in V$, and

$\|v - w\|$ = the distance between v and w in V.

The Cauchy-Schwarz inequality holds, since the proof only uses the properties of IPS.

Then, $\forall v, w \in V$, the angle $\theta_{v,w}$ between v and w is defined by the formula

$$\cos(\theta_{v,w}) = \frac{(v,w)}{(\|v\|)(\|w\|)}$$

and orthogonality is defined by $v \perp w$ when $(v,w)=0$.

Ex: Let V have basis $S=\{v_1, v_2\}$ and let ${}_S M_S = [(v_i, v_j)] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Then (v,w) is determined as follows. Let $[v]_S = \begin{bmatrix} a \\ b \end{bmatrix}$, $[w]_S = \begin{bmatrix} c \\ d \end{bmatrix}$ so $v = av_1 + bv_2$ and $w = cv_1 + dv_2$, so

$(v, w) = (av_1 + bv_2, cv_1 + dv_2) =$

$ac(v_1, v_1) + ad(v_1, v_2) + bc(v_2, v_1) + bd(v_2, v_2)$

$= 2ac - ad - bc + 2bd$. Also note:

$[v]_S^T \, {}_S M_S [w]_S = [a \ b] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$

$= [(2a-b) \ (-a+2b)] \begin{bmatrix} c \\ d \end{bmatrix} = [(2a-b)c + (-a+2b)d]$

$= [2ac - bc - ad + 2bd] \in \mathbb{R}^1_1 = \mathbb{R}$.

It is easy to check that this function is bilinear and symmetric. Is it pos. def.?

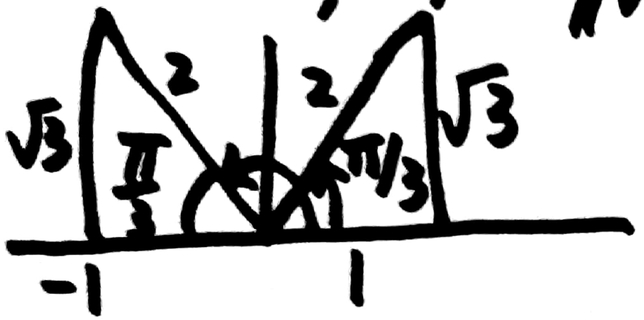
$(v, v) = 2a^2 - 2ab + 2b^2 = a^2 + (a^2 - 2ab + b^2) + b^2$

$= a^2 + (a-b)^2 + b^2 \geq 0$ and $(v, v) = 0$ iff

$0 = a = a-b = b$ iff $v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So it is pos. def.

From $_S M_S$ we have $\|v_i\| = \sqrt{(v_i, v_i)} = \sqrt{2}$, $i=1,2$ $\boxed{331}$

$$\cos(\theta_{v_1,v_2}) = \frac{(v_1, v_2)}{\|v_1\|\,\|v_2\|} = \frac{-1}{\sqrt{2}\sqrt{2}} = \frac{-1}{2} \text{ so } \theta_{v_1,v_2} = \frac{2\pi}{3}$$



For any V with basis $S = \{v_1, \dots, v_n\}$ and any $M \in \mathbb{R}^n_n$, <u>bilinear form</u> associated with M is $(v, w) = [v]_S^T M [w]_S$ so that $M = [(v_i, v_j)]$.

<u>Th</u>: $(v, w) = (w, v)$ iff $M = M^T$.

<u>Pf</u>. $(v, w) = (w, v)$ iff $[v]_S^T M [w]_S = [w]_S^T M [v]_S$

but these are 1×1 matrices, so equal to transpose: $[v]_S^T M [w]_S = ([w]_S^T M [v]_S)^T$ 332

$= [v]_S^T M^T [w]_S$. These are equal $\forall v, w \in V$ iff they are equal for all $v = v_i, w = v_j \in S$, that is, $e_i^T M e_j = e_i^T M^T e_j$, the (i,j)-entry of M equals the (i,j)-entry of M^T, so $M = M^T$. $\square$

What does it mean for this $(\cdot,\cdot)$ to be pos. def.? $\forall v \in V$, need $(v,v) \geq 0$ and $(v,v) = 0$ iff $v = \theta_V$. That means $\forall X = [v]_S \in \mathbb{R}^n$

$X^T M X \geq 0$ and $X^T M X = 0$ iff $X = 0_1^n$.

Def. Say real symm. $M \in \mathbb{R}_n^n$ is pos. def. when that condition above holds, $X^T M X \geq 0$, $\forall X \in \mathbb{R}^n$, ...

Function space examples: Let $a<b$ in $\mathbb{R}$.

Let $V=\{f:[a,b]\to\mathbb{R} \mid f \text{ is continuous}\}$ $= C[a,b]$. On this real vector space define

$$(f,g)=\int_a^b f(t)g(t)\,dt.$$

This is clearly bilinear and symmetric.

$(f,f)=\int_a^b f(t)^2\,dt \geq 0$ since $f(t)^2 \geq 0$ so the area under the curve $y=f(t)^2$ is ≥ 0 $(a\leq t\leq b)$. If $(f,f)=0$ that area is 0, forcing $f(t)=0$ for all $a\leq t\leq b$ (continuity used).

Th. If V is an IPS with inner product $(\cdot,\cdot)$ and $W \leq V$ is any subspace, then W is itself an IPS with $(\cdot,\cdot)$ as inner product. 334

Pf. The required properties of $(\cdot,\cdot)$ on W are inherited from V. $\square$

Application: Let $P[t] = \mathbb{R}[t]$ be the v.s. of all polynomials in t with real coeff's. Then all polys are continuous functions on $\mathbb{R}$, so for any $a<b$ in $\mathbb{R}$, they are a subsp. of $C[a,b]$ when restricted to domain $[a,b]$. The same holds for the finite dim'l vector space of polys. of degree $\leq m$, $P_m[t]$.

EX. For $P_2[t] = \{a_2t^2 + a_1t + a_0 \mid a_i \in \mathbb{R}\}$ considered as functions on $[0,1]$, define the bilinear, symm., pos. def. form:

$$(f(t), g(t)) = \int_0^1 f(t)g(t)\,dt.$$

For std. basis $S = \{\underset{f_1}{1}, \underset{f_2}{t}, \underset{f_3}{t^2}\}$ find the matrix of this form, ${}_SM_S = [(f_i, f_j)]$.

Solution: Since the basis polynomials are of the form t^i, $i = 0,1,2$, we find any entry by

$$(t^i, t^j) = \int_0^1 t^i t^j\,dt = \int_0^1 t^{i+j}\,dt = \left.\frac{t^{i+j+1}}{i+j+1}\right]_{t=0}^{t=1}$$

$$= \frac{1}{i+j+1} \text{ for } 0 \le i,j \le 2.$$

Thus, ${}_S M_S = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$.

<u>Exercise:</u> Using this inner product, do the Gram-Schmidt process on ordered basis S to get an orthogonal basis T for $P_2[t]$.

<u>Exercise.</u> Considering polynomials in $P_2[t]$ as functions on $[-1, 1]$, use the inner prod. $(f, g) = \int_{-1}^{1} f(t)g(t)\,dt$ instead of the $\int_0^1$ and redo the example above. How does ${}_S M_S$ change? How does G-S process change?

Suppose V is an IPS with basis $S = \{v_1, \dots, v_n\}$ and ${}_S M_S = [(v_i, v_j)]$, and basis $T = \{w_1, \dots, w_n\}$ with ${}_T M_T = [(w_i, w_j)]$. How are ${}_S M_S$ and ${}_T M_T$ related?

Know $(v, w) = [v]_S^{Tr} \, {}_S M_S [w]_S$ and

$$(v, w) = [v]_T^{Tr} \, {}_T M_T [w]_T \quad , \forall v, w \in V$$

and transition matrix ${}_S P_T$ satisfies

${}_S P_T [v]_T = [v]_S$, $\forall v \in V$. So, letting $P = {}_S P_T$,

$$\left(P [v]_T \right)^{Tr} {}_S M_S \left(P [w]_T \right) = [v]_T^{Tr} \, {}_T M_T [w]_T$$

$$= [v]_T^{Tr} \left(P^{Tr} \, {}_S M_S P \right) [w]_T$$

As usual, using $v = w_i$ and $w = w_j$ in T/338 gives $[v]_T = [w_i]_T = e_i$ and $[w]_T = [w_j]_T = e_j$ std. basis vectors in $\mathbb{R}^n$, so

$$e_i^{Tr}\left(P^{Tr}\,{}_S M_S\, P\right) e_j = e_i^{Tr}\,{}_T M_T\, e_j \quad \text{for } 1 \le i, j \le n$$

says the (i,j)-entries of $P^{Tr}\,{}_S M_S\, P$ and ${}_T M_T$ are equal, so ${}_T M_T = P^{Tr}\,{}_S M_S\, P$.

Def. For $A, B \in \mathbb{R}^n_n$, say A is congruent to B when $\exists\, P \in \mathbb{R}^n_n$ invertible such that $B = P^{Tr} A P$.

Thm: Congruence is an equivalence relation on $\mathbb{R}^n_n$.

<u>Th.</u> Let V be an IPS with bases S and T, ${}_SM_S$ and ${}_TM_T$ the matrices representing $(\cdot,\cdot)$ w.r.t. bases S and T, respectively. Then ${}_TM_T = P^{Tr}\,{}_SM_S\,P$ for $P = {}_SP_T$ transition matrix from T to S shows ${}_SM_S$ and ${}_TM_T$ are congruent.

This brings up the question: What choice of basis T makes ${}_TM_T$ "nicest"? Is there a "canonical form" for matrices representing an inner product (or for a more general bilinear form)?

<u>Answer</u>: We know ${}_SM_S$ is symmetric, so we showed it can be diagonalized orthogonally,

that is, using G-S process we can convert

S into an orthonormal basis T of e-vectors for ${}_S M_S$, so $P = {}_S P_T$ is an orthogonal matrix, $P^{-1} = P^{Tr}$, and then ${}_T M_T = P^{Tr}\, {}_S M_S\, P$ is diagonal with the e-values (all real) on the diagonal. If $\lambda \in \mathbb{R}$ is an e-value and $X = [w_i]_T \in \mathbb{R}^n$ is a corresponding e-vector, then

$$[w_i]_T^{Tr}\, {}_T M_T\, [w_i]_T = X^{Tr}\, {}_T M_T\, X$$

$$= X^{Tr}(\lambda X) = \lambda(X^{Tr} X) = \lambda\left(\sum_{i=1}^{n} x_i^2\right) \text{ for } X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

This must be > 0 (pos. def.) so $\lambda > 0$.