What can we say about general bilinear forms? Let V have basis $S = \{v_1, \dots, v_n\}$, F any field.

① ${}_SM_S = [f(v_i, v_j)]$ determines bil. form $f_{{}_SM_S} : V \times V \to F$ by $f_{{}_SM_S}(v, w) = [v]_S^{Tr} \, {}_SM_S [w]_S$.

② If $T = \{w_1, \dots, w_n\}$ is another basis of V, then ${}_TM_T = [f(w_i, w_j)]$ represents the same form f,
$f_{{}_TM_T}(v, w) = f_{{}_SM_S}(v, w)$ iff ${}_TM_T = P^{Tr} \, {}_SM_S \, P$
for invertible transition matrix $P = {}_SP_T$.

③ In that case, $\text{rank}({}_SM_S) = \text{rank}({}_TM_T)$ is called the rank of the form, $\text{rank}(f)$.

④ **Def.** If $f: V \times V \to F$ is any bil. form, $\lfloor 342$
say f is <u>non-degenerate</u> if $\operatorname{rank}(f) = \dim(V)$,
<u>degenerate</u> if $\operatorname{rank}(f) < \dim(V)$.

⑤ **Def.** Bil. form f on V is called <u>alternating</u>
if $f(v,v) = 0$, $\forall v \in V$; called <u>skew-symmetric</u>
if $f(u,v) = -f(v,u)$, $\forall u, v \in V$.
Note these are equivalent when $\operatorname{char}(F) \neq 2$,
that is, when $1+1 \neq 0$ in F.

⑥ <u>Th.</u> Suppose f is alternating bil. form on V.
Then V has a basis T such that ${}_T M_T$ of f is

$$\operatorname{diag}\Big(\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{m\text{-times}}, [0], \ldots, [0]\Big) \in F_n^n, \quad n = \dim(V)$$

so $2m = \operatorname{rank}(f)$.

⑦ **Th.** If f is a symm. bil. form on V and char$(F) \neq 2$ then V has a basis T s.t. matrix of f w.r.t. T is diagonal, ${}_T M_T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

Note: an algorithm to diagonalize any $A = A^T \in F_n^n$ is presented on p. 363-364 of our textbook, along with an example.

⑧ If $f_1, f_2 : V \times V \to F$ are bil. forms, then any lin. comb. $(a_1 f_1 + a_2 f_2) : V \times V \to F$ is also bil. The set of all bil. forms on V forms a vector space, $B(V)$, over F, isomorphic to F_n^n. Take any basis of V, S, and map $f \in B(V)$ to the matrix ${}_S M_S \in F_n^n$ representing f w.r.t. S. That map is a linear bijection.

Quadratic forms:

There are various definitions of a quadratic form on a vector space V over F.

Def 1. Function $q: V \to F$ is a quad. form on V if $q(v) = f(v,v)$ for some symm. bil. form f on V. Given q, can recover f by polarization

$$f(u,v) = \frac{1}{2}\Big(q(u+v) - q(u) - q(v)\Big).$$ Need $2 \neq 0$.

Check: $q(u+v) - q(u) - q(v) =$

$f(u+v, u+v) - f(u,u) - f(v,v) =$

$f(u,u) + f(u,v) + f(v,u) + f(v,v) - f(u,u) - f(v,v)$

$= 2f(u,v)$ since f is symm.

Note: $\forall a \in F$, $q(av) = f(av, av) = a^2 f(v,v) = a^2 q(v)$. This could be used to define a quad. form on V.

Def. A quad. form in n variables, $x_1, \ldots, x_n$ is a polynomial $q(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i^2 + \sum_{1 \le i < j \le n} d_{ij} x_i x_j$

If $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in F^n$ and $A = [a_{ij}] \in F_n^n$ where $a_{ii} = c_i$ and $a_{ij} = a_{ji} = \frac{1}{2} d_{ij}$ for $1 \le i < j \le n$ then $q(x_1, \ldots, x_n) = q(X) = X^{Tr} A X$.

Note. For $F = \mathbb{R}$, $V = \mathbb{R}^3$, $\{X \in \mathbb{R}^3 \mid q(X) = \text{constant}\}$ can be a surface; sphere, ellipsoid, paraboloid, hyperboloid, etc. "quadratic surface".

Th. (Sylvester's Law of Inertia)
Let f be a sym. bil. form on a real vector space V. Then V has a basis T s.t. matrix ${}_T M_T$ representing f is diagonal. Any other diag. matrix representing f has the same number, p, of positive entries, and the same number, n, of negative entries, so it has the same number z, of zero entries, and $p+n+z=\dim(V)$. $\operatorname{rank}(f)=p+n$, $\operatorname{sig}(f)=p-n$.

Def. f is pos. def. when $f(v,v)>0$ for $v\neq\theta$. f is pos. semidefinite if $f(v,v)\geq 0$ $\forall v\in V$.

We have seen that any $A = A^T \in \mathbb{R}^n_n$ can be orthog. diagonalized, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = P^T A P$ for $P^T = P^{-1}$, $P = {}_S P_T$.

If $Q = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}$ is invertible, $Q^T D Q =$

$\begin{bmatrix} \lambda_1 b_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n b_n^2 \end{bmatrix} = (PQ)^T A (PQ)$ is also congruent to A. Since each b_i can be chosen to be any nonzero real number, let $b_i = \dfrac{1}{\sqrt{|\lambda_i|}}$ if $\lambda_i \neq 0$, $b_i = 1$ if $\lambda_i = 0$. Then

$$\lambda_i b_i^2 = \begin{cases} \dfrac{\lambda_i}{|\lambda_i|} & \text{if } \lambda_i \neq 0 \\ 0 & \text{if } \lambda_i = 0 \end{cases} = \begin{cases} 1 & \text{if } \lambda_i > 0 \\ -1 & \text{if } \lambda_i < 0 \\ 0 & \text{if } \lambda_i = 0. \end{cases}$$

Cor. Any real symm. matrix is congruent to a unique diagonal matrix $D = \mathrm{diag}(I_p, -I_n, O_z^z)$ with rank $p+n$.

Cor. Any real quad. form q has a unique representation of the "diagonal" form $q(x_1, \ldots, x_m) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+n}^2$ where $\mathrm{rank}(q) = p+n$.

Hermitian forms on a complex vectorspace

$f: V \times V \to \mathbb{C}$ is called a Hermitian form if

① f is linear in first variable,

② $f(u,v) = \overline{f(v,u)}$, so f is conjugate linear in the second variable.

These imply $f(v,v) = \overline{f(v,v)} \in \mathbb{R}$, $\forall v \in V$, and the associated quad. form $q(v) = f(v,v)$ is called the Hermitian quad. form assoc. to f. There is a polarization formula for $f(u,v)$ in terms of q but it is more complicated than the one for real quad. forms. (See p. 366, last line).

See Th. 12.7 (p. 367) for diagonalization of Hermitian forms, $\operatorname{rank}(f)$, $\operatorname{sig}(f)$, p and n as before.

Positive definite Hermitian forms are important in physics.