Dual space (linear functionals):

<u>Def</u>. For any vector space V over field F, the <u>dual space</u> of V is $V^* = \mathrm{Lin}(V, F)$.

<u>Examples</u>. Let $V = F^n$ with std. basis $S = \{e_1, \dots, e_n\}$ so $\forall v \in F^n$, $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^{n} a_i e_i$.

Let $\pi_i : F^n \to F$ be the i^{th} projection map s.t. $\pi_i(v) = a_i$. Then $\pi_i \in V^*$ for each $1 \le i \le n$.

<u>Claim</u>: $S^* = \{\pi_1, \dots, \pi_n\}$ is a basis of V^* s.t. $\pi_i(e_j) = \delta_{ij}$ for $1 \le i, j \le n$.

<u>Pf</u>. $\forall f \in V^*$, f is determined by its values

$f(e_j) = b_j \in F$ on the basis S. But

$$\left(\sum_{i=1}^{n} b_i \pi_i\right)(e_j) = \sum_{i=1}^{n} b_i \pi_i(e_j) = \sum_{i=1}^{n} b_i \delta_{ij} = b_j$$

for $1 \le j \le n$ shows $f = \sum_{i=1}^{n} b_i \pi_i$ so $V^* = \langle S^* \rangle$.

If $\sum_{i=1}^{n} b_i \pi_i = \theta_{V^*}$ then $\sum_{i=1}^{n} b_i \pi_i(e_j) = 0 \in F$

says $b_j = 0$ for $1 \le j \le n$, so S^* is indep. $\square$

<u>Note</u>: In that example, $\dim(V^*) = \dim(V)$

Generally, if V is any fin. dim'l v.s. with basis $S = \{v_1, \dots, v_n\}$, $f \in V^*$ is determined by its values on S, $f(v_j) = b_j \in F$ for $1 \le j \le n$.

Define the <u>dual</u> <u>basis</u> $S^* = \{f_1, \dots, f_n\}$ of V^* where $f_i(v_j) = \delta_{ij}$, $1 \le i, j \le n$, determining each f_i. The proof just given works in this situation and shows $\langle S^* \rangle = V^*$ and S^* is indep, justifying the name "dual basis."

<u>Example</u>: Let $V = \mathbb{R}[t] = Poly[t]$ be the real v.s. of polynomials in variable t. The functional $J_{a,b}: V \to \mathbb{R}$ defined by $J_{a,b}(p(t)) = \int_a^b p(t)\,dt$ is a lin. map for any choice of $a < b$ in $\mathbb{R}$.

<u>Ex</u>: The trace, $Tr: F_n^n \to F$, $Tr[a_{ij}] = \sum_{i=1}^n a_{ii}$ is lin. fcn'l.

Note. In the first example where $V = F^n$ if $v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $\pi_i(v) = a_i = e_i^{Tr} v = [0, \ldots, 0, 1, 0, \ldots, 0] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ where $e_i^{Tr} \in F_n$. So the map $\phi : V^* \to F_n$ where $\phi(\pi_i) = e_i^{Tr}$, is an isomorphism s.t. $\phi\left(\sum_{i=1}^{n} b_i \pi_i\right) = \sum_{i=1}^{n} b_i e_i^{Tr} = [b_1, \ldots, b_n]$. We can think of $b = [b_1, \ldots, b_n] \in F_n$ as providing the linear functional $f_b : F^n \to F$ by $f_b(v) = \underset{(1\times n)}{b} \underset{(n\times 1)}{v}$ (matrix mult.).

A linear function of n variables is of the form $f(x_1, \ldots, x_n) = b_1 x_1 + \cdots + b_n x_n$ for some constants $b_i \in F$.

Problem: Let $V = \mathbb{R}^2$, $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ a basis of V. Find the dual basis $S^* = \{f_1, f_2\}$. 354

Solution. Find $f_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11} x_1 + a_{12} x_2$ and $f_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{21} x_1 + a_{22} x_2$ such that $f_i(v_j) = \delta_{ij}$ for $1 \le i, j \le 2$. That is,

$$\left.\begin{array}{l} f_1(v_1) = a_{11}(1) + a_{12}(2) = 1 \\ f_1(v_2) = a_{11}(1) + a_{12}(3) = 0 \\ f_2(v_1) = a_{21}(1) + a_{22}(2) = 0 \\ f_2(v_2) = a_{21}(1) + a_{22}(3) = 1 \end{array}\right\} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

So $f_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1 - x_2$, $f_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2x_1 + x_2$

Check that $f_i(v_j) = \delta_{ij}$.

Th. Let $S=\{v_1,\dots,v_n\}$ be a basis of V and let $S^*=\{f_1,\dots,f_n\}$ be the dual basis of V^*. Then

(1) $\forall v\in V$, $v=\sum_{i=1}^{n} f_i(v)\, v_i$ so $[v]_S=\begin{bmatrix} f_1(v)\\ \vdots \\ f_n(v)\end{bmatrix}$

(2) $\forall f\in V^*$, $f=\sum_{i=1}^{n} f(v_i)\, f_i$ so $[f]_{S^*}=\begin{bmatrix} f(v_1)\\ \vdots \\ f(v_n)\end{bmatrix}$

Pf. (1) If $v=\sum_{i=1}^{n} x_i v_i$ then for $1\le j\le n$, we have

$$f_j(v)=\sum_{i=1}^{n} x_i f_j(v_i)=\sum_{i=1}^{n} x_i \delta_{ij}=x_j.$$

(2) If $f=\sum_{i=1}^{n} y_i f_i$ then for $1\le j\le n$, we have

$$f(v_j)=\sum_{i=1}^{n} y_i f_i(v_j)=\sum_{i=1}^{n} y_i\delta_{ij}=y_j. \quad \square$$

<u>Th</u>. Let $S = \{v_1, \dots, v_n\}$ and $T = \{w_1, \dots, w_n\}$ be bases of V with ${}_S P_T$ transition matrix from T to S s.t. ${}_S P_T [v]_T = [v]_S$. Let $S^* = \{f_1, \dots, f_n\}$ and $T^* = \{g_1, \dots, g_n\}$ be the dual bases of V^* with ${}_{S^*}P_{T^*}$ transition matrix from T^* to S^* s.t. ${}_{S^*}P_{T^*} [f]_{T^*} = [f]_{S^*}$.

Then ${}_{S^*}P_{T^*} = \left({}_S P_T^{-1}\right)^{Tr} = \left({}_T P_S\right)^{Tr}$.

<u>Pf</u>. We know $Col_j\left({}_{S^*}P_{T^*}\right) = [g_j]_{S^*} = \begin{bmatrix} g_j(v_1) \\ \vdots \\ g_j(v_n) \end{bmatrix}$ by (2) of last Theorem. So

$${}_{S^*}P_{T^*} = \begin{bmatrix} g_1(v_1) & g_2(v_1) & \cdots & g_n(v_1) \\ \vdots & \vdots & & \vdots \\ g_1(v_n) & g_2(v_n) & \cdots & g_n(v_n) \end{bmatrix} = [g_j(v_i)].$$

Also, $\mathrm{Col}_i({}_TP_S) = [v_i]_T = \begin{bmatrix} g_1(v_i) \\ \vdots \\ g_n(v_i) \end{bmatrix}$ by (1) of last Theorem. So

$$ {}_TP_S = \begin{bmatrix} g_1(v_1) & g_1(v_2) & \cdots & g_1(v_n) \\ \vdots & \vdots & & \vdots \\ g_n(v_1) & g_n(v_2) & \cdots & g_n(v_n) \end{bmatrix} = [g_i(v_j)]. $$

Thus, ${}_{S^*}P_{T^*} = ({}_TP_S)^{Tr} = ({}_SP_T^{-1})^{Tr}$. □

Second dual space:

Since we can take the dual of any vector space, V^* has a dual space, $V^{**} = (V^*)^*$, the "double dual" of V. Of course, $V^{**} = \mathrm{Lin}(V^*, F)$ by def. but there is a natural map from V to V^{**}, $v \longmapsto \hat{v}$ where $\hat{v} \in V^{**}$ is defined by $\hat{v}(f) = f(v)$, $\forall f \in V^*$.

Check that $\hat{v}$ is linear: $\forall f, g \in V^*$, $\forall a, b \in F$,
$\hat{v}(af+bg) = (af+bg)(v) = af(v) + bg(v) =$
$a\hat{v}(f) + b\hat{v}(g)$.

Th. $\forall v, w \in V$, if $\hat{v} = \hat{w}$ then $v = w$, so the map $\hat{\ } : V \to V^{**}$ is injective.

Pf. Suppose $\hat{v} = \hat{w}$ in V^{**}. Then $\forall f \in V^*$,
$f(v) = \hat{v}(f) = \hat{w}(f) = f(w)$ so $f(v-w) = 0$.
If $v \neq w$ then $\exists f \in V^*$ s.t. $f(v-w) \neq 0$, so $v = w$. $\square$

Th: If $\dim(V) = n < \infty$ (finite) then the map $\hat{\ } : V \to V^{**}$ is surjective, so it is an isomorphism.

Pf. We know $\dim(V^*) = \dim(V) = n$ is finite, so $\dim(V^{**}) = \dim(V^*) = \dim(V) = n$. The map $\hat{\ }$ is injective between V and V^{**} of the same dimension, so it is also onto. $\square$