Annihilators:

Let $S \subseteq V$ be any subset of v.s. V.

Def. $Ann(S) = \{f \in V^* \mid f(s)=0, \forall s \in S\} := S^\circ$

Th $S^\circ \leq V^*$ and $S^\circ = \langle S \rangle^\circ$

Pf. $\theta_{V^*} \in V^*$ is defined by $\theta_{V^*}(v)=0, \forall v \in V$, so $\theta_{V^*} \in S^\circ$. If $f, g \in S^\circ$ then $\forall a, b \in F$, $(af+bg)(s) = af(s)+bg(s) = 0+0=0$ so $af+bg \in S^\circ$. Thus $S^\circ \leq V^*$. If $v \in \langle S \rangle$ then $v = \sum_{i=1}^{m} a_i s_i$ for some $s_i \in S$ and $a_i \in F$. $\forall f \in S^\circ$, $f(v) = \sum a_i f(s_i) = 0$ so $f \in \langle S \rangle^\circ$, showing $S^\circ \subseteq \langle S \rangle^\circ$.

If $g \in \langle S \rangle^\circ$ then $g(s)=0, \forall s \in S$, since $s \in \langle S \rangle$, so $\langle S \rangle^\circ \subseteq S^\circ$ giving equality. $\square$

Th. Let $\dim(V)$ be finite and $W \leq V$. Then

(1) $\dim(W) + \dim(W^\circ) = \dim(V)$,

(2) $(W^\circ)^\circ = W$.

Pf. See Problem 11.11 in our textbook.

Transpose of a lin. map: $\forall L : V \to \mathcal{U}$ and $\forall f \in \mathcal{U}^*$ have $f \circ L \in V^*$.

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[dr, "f \circ L"'] & \mathcal{U} \arrow[d, "f"] \\ & F \end{tikzcd}$$

This defines $L^* : \mathcal{U}^* \to V^*$ by $L^*(f) = f \circ L$ and $\forall v \in V$, $(L^*(f))(v) = f(L(v))$. In textbook, L^* is denoted by L', called the <u>transpose</u> of L.

Th. $L^*: U^* \to V^*$ is a linear map.

Pf. $\forall f, g \in U^*$, $\forall a, b \in F$,

$$L^*(af+bg) = (af+bg)\circ L = a(f\circ L) + b(g\circ L)$$
$$= a L^*(f) + b L^*(g). \ \square$$

Th. Let $L: V \to U$, $S = \{v_1, \dots, v_n\}$ a basis of V, $T = \{u_1, \dots, u_m\}$ a basis of U, $A = {}_T[L]_S \in F^m_n$.
Let $S^* = \{f_1, \dots, f_n\}$ and $T^* = \{g_1, \dots, g_m\}$ be dual bases of V^* and U^*, respectively, and let $B = {}_{S^*}[L^*]_{T^*} \in F^n_m$ represent $L^*: U^* \to V^*$ w.r.t. T^* and S^*. Then $B = A^{Tr}$.

Pf. See problem 11.16 in our textbook.

Linear operators on an I.P.S. ($F = \mathbb{R}$ or $\mathbb{C}$)

For $V = \mathbb{R}^n$, std. inner product $\langle u, v \rangle = u^{Tr} v$.
For $V = \mathbb{C}^n$, std. inner product $\langle u, v \rangle = u^{Tr} \overline{v}$.
Hermitian conjugate of $A \in \mathbb{C}^n_n$ is now to be denoted by $A^* = \overline{A}^T$.

<u>Def.</u> For lin. map (operator) $L : V \to V$ (I.P.S.) its <u>adjoint</u> is $L^* : V \to V$ s.t. $\langle L(u), v \rangle = \langle u, L^*(v) \rangle$ $\forall u, v \in V$.

<u>Ex.</u> For $L = L_A : \mathbb{R}^n \to \mathbb{R}^n$, $A \in \mathbb{R}^n_n$, we have

$$\langle L_A(u), v \rangle = \langle Au, v \rangle = (Au)^T v = u^T (A^T v) = \langle u, L_{A^T}(v) \rangle$$
$$= \langle u, A^T v \rangle, \ \forall u, v \in \mathbb{R}^n.$$

So $L_A^* = L_{A^T}$.

For $A \in \mathbb{C}^n_n$, $L = L_A : \mathbb{C}^n \to \mathbb{C}^n$ we have

$$\langle L_A(u), v \rangle = (Au)^T \bar{v} = u^T (A^T \bar{v}) = u^T \overline{(\bar{A}^T v)}$$

$$= \langle u, \bar{A}^T v \rangle = \langle u, L_{A^*}(v) \rangle, \ \forall u, v \in \mathbb{C}^n, \text{ so}$$

$$(L_A)^* = L_{A^*}.$$

Sometimes we say the adjoint of matrix A is A^*.

If V is an I.P.S. (real or complex) with basis $S = \{v_1, \dots, v_n\}$ and ${}_S M_S = [\langle v_i, v_j \rangle]$ then we know $\forall u, v \in V$, $\langle u, v \rangle = [u]_S^{Tr} \, {}_S M_S \, \overline{[v]_S}$ where the "bar" has no effect if $F = \mathbb{R}$.

$L : V \to V$ has adjoint $L^* : V \to V$ s.t.
$\forall u, v \in V$, $\langle L(u), v \rangle = \langle u, L^*(v) \rangle$ iff $A = {}_s[L]_s$, $B = {}_s[L^*]_s$

satisfy $\quad [L(u)]_s^{Tr}\,{}_sM_s\,\overline{[v]_s} = [u]_s^{Tr}\,{}_sM_s\,\overline{[L^*(v)]_s}$

iff $\quad ({}_s[L]_s[u]_s)^{Tr}\,{}_sM_s\,\overline{[v]_s} = [u]_s^{Tr}\,{}_sM_s\,\overline{({}_s[L^*]_s[v]_s)}$

iff $\quad [u]_s^{Tr}(A^{Tr}\,{}_sM_s)\overline{[v]_s} = [u]_s^{Tr}({}_sM_s\overline{B})\overline{[v]_s}$

So with $u = v_i$, $v = v_j \in S$ get $[u]_s = e_i$, $[v]_s = e_j$

get conditions $\quad e_i^{Tr}(A^{Tr}\,{}_sM_s)\overline{e_j} = e_i^{Tr}({}_sM_s\overline{B})\overline{e_j}$ so

$A^{Tr}\,{}_sM_s = {}_sM_s\overline{B}$ since their entries match.

In general, all we know about ${}_sM_s$ is that it is Hermitian and positive definite since it represents an inner product on V, so ${}_sM_s = \overline{{}_sM_s}^{Tr}$.

Since ${}_sM_s$ is invertible, we get

${}_sM_s^{-1} A^{Tr} {}_sM_s = \bar{B}$ so $B = {}_s\bar{M}_s^{-1} \bar{A}^{Tr} {}_s\bar{M}_s$

$= ({}_sM_s^{Tr})^{-1} A^* {}_sM_s^{Tr} = ({}_sM_s^{-1})^{Tr} \bar{A}^{Tr} {}_sM_s^{Tr}$

$= ({}_sM_s \bar{A} {}_sM_s^{-1})^{Tr}$. The point is, B is uniquely determined by A so L^* is uniquely determined by L.

<u>Th</u>. For any $L : V \to V$ I.P.S. there is a unique $L^* : V \to V$ s.t. $\langle L(u), v\rangle = \langle u, L^*(v)\rangle$.

How does this simplify when basis S is orthonormal?

S is orthonormal iff ${}_SM_S = [\langle v_i, v_j\rangle] = I_n$

so the condition relating A and B becomes

$A^{Tr} = \overline{B}$, that is, $B = \overline{A}^{Tr} = A^*$.

<u>Th</u>. If $L: V \to V$ I.P.S. and S is an orthonormal basis of V, so ${}_SM_S = [\langle v_i, v_j\rangle] = I_n$, then

$\langle u, v\rangle = [u]_S^{Tr}\overline{[v]_S}$ and ${}_S[L^*]_S = {}_S[L]_S^*$.

<u>Th</u>. For $L, L_1, L_2 : V \to V$ I.P.S. and $a \in F$ we have

(1) $(L_1 + L_2)^* = L_1^* + L_2^*$, (2) $(aL)^* = \bar{a}L^*$,

(3) $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$, (4) $(L^*)^* = L$.

<u>Pf</u>. Problem 13.5.

Linear Functionals and I.P.S.

If V is an I.P.S then define the map $\hat{}: V \to V^*$ (dual space) by $\hat{u}(v) = \langle v, u \rangle$ for $u, v \in V$. $\hat{u}$ is clearly linear in v since $\langle v, u \rangle$ is linear in its first input. Also $(u_1+u_2)\hat{}\,(v) = \langle v, u_1+u_2 \rangle = \langle v, u_1 \rangle + \langle v, u_2 \rangle = \hat{u}_1(v) + \hat{u}_2(v)$ so $(u_1+u_2)\hat{} = \hat{u}_1 + \hat{u}_2$. But for $a \in F$ $(au)\hat{}\,(v) = \langle v, au \rangle = \bar{a}\langle v, u \rangle = \bar{a}\,\hat{u}(v)$ so $(au)\hat{} = \bar{a}\,\hat{u}$. If $F = \mathbb{R}$ then $\hat{}: V \to V^*$ is linear but if $F = \mathbb{C}$ then it is conjugate linear.

If $\dim(V)$ is finite, $\dim(V^*) = \dim(V)$.

If $\hat{u}_1 = \hat{u}_2$ then $\forall v \in V$, $\langle v, u_1 \rangle = \langle v, u_2 \rangle$ so

$\langle v, u_1 - u_2 \rangle = 0$ for all $v \in V$ including $v = u_1 - u_2$, so $\langle u_1 - u_2, u_1 - u_2 \rangle = 0$ which gives $u_1 = u_2$ since $\langle \cdot, \cdot \rangle$ is positive definite.

<u>Th</u>: For any I.P.S. V, the map $\hat{\ }: V \to V^*$ is injective. When $\dim(V) = n$ is finite, $\hat{\ }$ is also surjective, so bijective, that is, $\forall f \in V^*$, $\exists u \in V$ s.t. $f = \hat{u}$.

<u>Def.</u> $L: V \to V$ is called <u>self-adjoint</u> when $L = L^*$. A matrix $A \in \mathbb{C}^n_n$ is self-adjoint when $A = A^*$. So for $F = \mathbb{R}$, $A \in \mathbb{R}^n_n$, this means $A = A^{Tr}$ is symmetric. For $F = \mathbb{C}$, $A \in \mathbb{C}^n_n$, it says $A = \bar{A}^{Tr}$ is Hermition.