Note: For any $L : V \to V$ let $H = L^* \circ L$.
Then $H^* = (L^* L)^* = L^* L = H$ is self-adjoint.
Similar for matrices, $A^* A = B$ is Hermitian.
For $X \in \mathbb{C}^n$, $X^{Tr}(A^* A)\bar{X} = (X^{Tr} \bar{A}^{Tr})(A \bar{X})$
$= (\bar{A} X)^{Tr} \overline{(\bar{A} X)} = \langle \bar{A} X, \bar{A} X \rangle \geq 0$ and "=" iff
$\bar{A} X = 0_1^n \in \mathbb{C}^n$.

Th. If $A \in \mathbb{C}_n^n$ is invertible then $A^* A$ is positive definite Hermetian.

Def. $L : V \to V$ is called skew-adjoint when $L^* = -L$. For matrices $A^* = -A$ mean skew-Herm. in complex case, skew-symm. in real case, $A^{Tr} = -A$.

See Table 13.1 on p. 381 for analogies between certain subsets of $\mathbb{C}$ and certain subsets of operators in $End(V) = Lin(V, V)$, as well as analogies between their behavior under conjugation (in $\mathbb{C}$) and under adjoint $*$ in $End(V)$.

<u>Th.</u> Let $L : V \to V$ have e-value $\lambda \in F$ ($\mathbb{R}$ or $\mathbb{C}$)

(1) If $L^* = L^{-1}$ then $|\lambda| = 1$

(2) If $L^* = L$ then $\lambda \in \mathbb{R}$.

(3) If $L^* = -L$ then $\lambda \in i\mathbb{R}$ (pure imaginary)

(4) If $L = K^* K$ for invertible $K : V \to V$ then $0 < \lambda \in \mathbb{R}$.

<u>Pf.</u> See Page 381.

Th. Suppose $L : V \to V$ is self-adjoint, so $L = L^*$, and L has e-values $\lambda \neq \mu$ with e-vectors u, v so $L(u) = \lambda u$, $L(v) = \mu v$. Then $\langle u, v \rangle = 0$.

Pf. $\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle L(u), v \rangle = \langle u, L^*(v) \rangle$
$= \langle u, L(v) \rangle = \langle u, \mu v \rangle = \bar{\mu} \langle u, v \rangle = \mu \langle u, v \rangle$
since e-values $\lambda, \mu \in \mathbb{R}$ for $L = L^*$. Then $(\lambda - \mu) \langle u, v \rangle = 0$ gives $\langle u, v \rangle = 0$. $\square$

Recall: $L : V \to V$ (I.P.S) is called
orthogonal if $L^* = L^{-1}$ in case $F = \mathbb{R}$,
unitary if $L^* = L^{-1}$ in case $F = \mathbb{C}$.

The following Theorem requires $\dim(V)$ finite.

Th. The following are equivalent if $\dim(V)<\infty$

(1) $L^* = L^{-1}$, (2) $\langle L(u), L(v)\rangle = \langle u, v\rangle$, $\forall u, v \in V$,

(3) $\|L(v)\| = \|v\|$, $\forall v \in V$. **Pf.** Problem 13.10.

Note. As a counterexample when $\dim(V)=\infty$, let $V = \{(a_1, a_2, \dots) \mid a_i \in \mathbb{R}, \sum_{i=1}^{\infty} a_i^2 < \infty\}$ the l_2 Hilbert space from Section 7.3 (p. 231). $L: V \to V$ defined by $L(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$ satisfies (2) and (3) but is not surjective so not invertible.

Def. Say that an isomorphism $L: V \to W$ between I.P.S.'s s.t. $\langle v_1, v_2\rangle_V = \langle L(v_1), L(v_2)\rangle_W$ is an <u>isometry</u>.

Def. Let $1 \le n \in \mathbb{Z}$. Define

$$U(n) = \{A \in \mathbb{C}^n_n \mid A^* = A^{-1}\} = \{n \times n \text{ unitary matrices}\}$$

$$O(n) = \{A \in \mathbb{R}^n_n \mid A^T = A^{-1}\} = \{\text{orthogonal matrices}\}$$

Th. $U(n)$ and $O(n)$ are groups under matrix multiplication. **Pf.** Exercise.

Recall. These matrices are characterized by the property that their columns (rows) form an orthonormal basis of F^n (F_n) w.r.t. std. dot product.

Th. Let V be an I.P.S. with bases S and T.
(1) If S and T are orthonormal, then ${}_SP_T$ and ${}_TP_S$ are orthog. if $F=\mathbb{R}$, unitary if $F=\mathbb{C}$.

(2) If S is orthonormal and ${}_T P_S$ is orthog. if $F = \mathbb{R}$, unitary if $F = \mathbb{C}$, then T is orthonormal. <u>Pf.</u> Problem 13.12.

<u>Def.</u> Say $A, B \in \mathbb{C}^n_n$ are <u>unitarily equivalent</u> if $\exists\ P \in U(n)$ s.t. $B = P^* A P$. Say $A, B \in \mathbb{R}^n_n$ are <u>orthogonally equiv.</u> if $\exists\ P \in O(n)$ s.t. $B = P^T A P$.

<u>Def</u> For I.P.S. V an operator $L : V \to V$ is called <u>positive definite</u> if $L = K^* \circ K$ for some invertible $K : V \to V$. The operator L is <u>pos. semidefinite</u> if $L = K^* \circ K$ for any $K : V \to V$ not necessarily invertible.

Th. These conditions on $L: V \to V$ (I.P.S.) are equivalent:

(1) $L = K^2 = K \circ K$ for an invertible $K = K^*$,

(2) L is positive definite,

(3) $L = L^*$ and $\langle L(v), v \rangle > 0$, $\forall\, \theta \neq v \in V$.

There is a similar theorem for pos. semidef. L (see p. 384) with (3) modified to $\langle L(v), v \rangle \geq 0$.

Diagonalization and Canonical Forms in I.P.S.

Real I.P.S. Case:

Th. If $L = L^* : V \to V$ for $\dim(V) < \infty$ then V has a basis T which is orthonormal and consists of e-vectors for L, so ${}_T[L]_T$ is diagonal.

Th. (Analog of last Th. for matrices)
If $A = A^T \in \mathbb{R}^n_n$ then $\exists\ P \in \mathcal{O}(n)$ s.t.
$D = P^{-1}AP = P^TAP$ is diagonal.

Note: An orthog. operator may not be represented by a symmetric matrix, for example rotation by angle ϕ has rep'n
$L_\phi : \mathbb{R}^2 \to \mathbb{R}^2$
$${}_S[L_\phi]_S = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = R_\phi \text{ for std. basis } S.$$
While not diag-able, such orthog. operators do have a canonical form. (See Prob. 13.16).

Def. Let V be a real I.P.S. with I.P. $\langle \cdot, \cdot \rangle$.
$$\mathcal{O}(V) = \{ L : V \to V \mid \langle L(v), L(w) \rangle = \langle v, w \rangle,\ \forall v, w \in V \}.$$

<u>Th.</u> For $L \in \mathcal{O}(V)$, $\exists$ orthonormal basis T of V s.t. ${}_T[L]_T = \mathrm{diag}(I_s, -I_t, R_{\phi_1}, R_{\phi_2}, \dots, R_{\phi_r})$ for some $0 \le s, t, r \in \mathbb{Z}$ with $\dim(V) = s+t+2r$.

Complex I.P.S. Case:

<u>Def.</u> $L : V \to V$ is called <u>normal</u> if $LL^* = L^*L$ so L commutes with its adjoint. Matrix $A \in \mathbb{C}^n_n$ is called <u>normal</u> if $A^*A = AA^*$ so A commutes with $\bar{A}^{Tr} = A^*$.

<u>Th.</u> If $L : V \to V$ (fin. dim'l. I.P.S.) is normal then $\exists$ basis T orthonormal e-basis, so ${}_T[L]_T = D$ is diagonal.

<u>Th.</u> (Analog of last Theorem for matrices)
If $A \in \mathbb{C}_n^n$ is normal $(AA^* = A^*A)$ then
$\exists\, P \in \mathcal{U}(n)$ s.t. $D = P^{-1}AP = P^*AP$ is diag.

<u>Th.</u> For any $L : V \to V$ (fin. dim'l complex I.P.S.)
$\exists$ orthonormal basis T of V s.t. ${}_T[L]_T$ is upper triangular.

<u>Th.</u> For any $A \in \mathbb{C}_n^n$, $\exists\, P \in \mathcal{U}(n)$ s.t.
$B = P^{-1}AP = P^*AP$ is upper triangular.

<u>Th.</u> (Spectral Th.) Let $L : V \to V$ be normal if $F = \mathbb{C}$,
$L = L^*$ if $F = \mathbb{R}$, V a fin. dim'l I.P.S. Then for some
$1 \le r \in \mathbb{Z}$, $\exists\, E_i : V \to V$, $\exists\, \lambda_i \in F$, $1 \le i \le r$, such that
(1) $L = \sum \lambda_i E_i$ (2) $\sum E_i = I_V$ (3) $E_i^2 = E_i$ (4) $E_i E_j = 0$ for $i \ne j$.