Def. Let I be any index set and F any field. Define $\prod_{i\in I} F = \{f : I \to F\}$, the "direct product", with addition and scalar multiplication defined "componentwise", that is, $\forall f, g \in \prod_{i\in I} F = F^I$, $f+g \in F^I$ where $\forall i \in I$, $(f+g)(i) = f(i)+g(i)$ and $\forall a \in F$, $af \in F^I$ where $\forall i \in I$, $(af)(i) = a\cdot f(i)$.

Let $\theta = \theta_{F^I}$ be defined by $\theta(i) = 0$ so that $\forall f \in F^I$, $(f+\theta)(i) = f(i) + \theta(i) = f(i) + 0 = f(i)$ so $f + \theta = f$.

Th: $(F^I, +, \cdot, \theta)$ is a vector space.

Pf. Exercise.

Def. Let the direct sum

$\coprod_{i\in I} F = \bigoplus_{i\in I} F = \{f \in F^I \mid \text{supp}(f) \text{ is finite}\}$ where

$\text{supp}(f) = \{i \in I \mid f(i) \neq 0\}$.

Th. $\coprod_{i\in I} F \leq F^I$ (is a subspace of F^I).

These are "=" iff I is finite.

Def. For $i \in I$ suppose V_i is a vector space over F. Let $\prod_{i\in I} V_i = \{f : I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i,\ \forall i \in I\}$

and $\forall f, g$ in it, let $(f+g)(i) = f(i) + g(i) \in V_i$

and $\forall a \in F$, let $(af)(i) = a \cdot f(i) \in V_i$, $\theta(i) = \theta_{V_i} \in V_i$.

Th: $\left(\prod_{i\in I} V_i, +, \cdot, \theta\right)$ is a vector space over F.

Def. Let the direct sum of $\{V_i \mid i\in I\}$ be

$$\coprod_{i\in I} V_i = \bigoplus_{i\in I} V_i = \left\{f \in \prod_{i\in I} V_i \;\middle|\; \mathrm{Supp}(f) \text{ is finite}\right\}$$

where $\mathrm{supp}(f) = \{i\in I \mid f(i) \neq \theta_{V_i}\}$.

Th: $\left(\coprod_{i\in I} V_i, +, \cdot, \theta\right)$ is a subspace of $\prod_{i\in I} V_i$.

("=" iff I is finite).