Def. For vector space V and $1 \le m \in \mathbb{Z}$ let $V^m = \prod_{i=1}^{m} V = \{f : I \to V\}$ where $I = \{1, 2, \dots, m\}$.

$\forall f \in V^m$, f is determined by the "m-tuple" $(f(1), f(2), \dots, f(m)) \in V \times V \times \dots \times V$ (m-times). So $V^m = \{(v_1, v_2, \dots, v_m) \mid v_i \in V, 1 \le i \le m\}$ is a vector space.

Def. Let V, U be vector spaces over field F and let $1 \le m \in \mathbb{Z}$. Say a map $L : V^m \to U$ is multilinear (m-linear) when it is linear in each of the entries v_i separately, that is,

$$L(\dots, \underset{\uparrow \text{fixed}}{}av_i + bv_i', \dots) = aL(v_1, \dots, v_i, \dots, v_m) + bL(v_1, \dots, v_i', \dots, v_m).$$

Def. Say that multilinear map $L: V^m \to U$ is <u>alternating</u> when $v_i = v_j$ for $1 \le i \ne j \le m$ implies $L(v_1, \dots, v_m) = 0$.

Note. If L is alternating then

$$L(v_1, \dots, v_i, \dots, v_j, \dots, v_m) = -L(v_1, \dots, v_j, \dots, v_i, \dots, v_m)$$

so L is "anti-symmetric" in each pair of entries.

Pf. $0 = L(v_1, \dots, \underbrace{v_i + v_j}_{i^{th}\text{ entry}}, \dots, \underbrace{v_i + v_j}_{j^{th}\text{ entry}}, \dots, v_m)$ (by multilinearity)

$= L(v_1, \dots, v_i, \dots, v_i, \dots, v_m) + L(v_1, \dots, v_i, \dots, v_j, \dots, v_m)$

$+ L(v_1, \dots, v_j, \dots, v_i, \dots, v_m) + L(v_1, \dots, v_j, \dots, v_j, \dots, v_m)$

$= 0 + L(v_1, \dots, v_i, \dots, v_j, \dots, v_m) + L(v_1, \dots, v_j, \dots, v_i, \dots, v_m) + 0$. □

Def. Let $V = F_n = \{(a_1, \dots, a_n) \mid a_j \in F,\ 1 \le j \le n\}$ and think of $V^m = (F_n)^m = \{(v_1, \dots, v_m) \mid v_i \in V,\ 1 \le i \le m\}$ for $v_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in V = F_n$. The map $Mat_n^m : V^m \longrightarrow F_n^m$ defined by $Mat_n^m((v_1, \dots, v_m)) = [a_{ij}]$ is an isomorphism. It has inverse map $Rows_n^m : F_n^m \to V^m$ where $Rows_n^m([a_{ij}]) = (Row_1(A), Row_2(A), \dots, Row_m(A))$ if $A = [a_{ij}]$.

We can apply these ideas to get a new viewpoint on $\det : F_n^n \to F$ as follows.

Let $V = F_n$ and for $(v_1,\dots,v_n) \in V^n$ with each $v_i = (a_{i1},\dots,a_{in}) \in F_n$, let $L : V^n \to F$ be defined by $L(v_1,\dots,v_n) = \det(\mathrm{Mat}_n^n(v_1,\dots,v_n)) = \det(A)$ where $v_i = \mathrm{Row}_i(A)$. Then we know from properties of $\det(A)$ that L is multilinear and alternating.

__Th.__ (Prob. 8.37, page 288) If $L : V^n \to F$ is multilinear, alternating and $L(e_1,\dots,e_n) = 1$ for std. basis vectors $\{e_1,\dots,e_n\}$ of F_n, then $L = \det \circ \mathrm{Mat}_n^n$.

A multilinear map is a generalization of bilinear maps on a Cartesian product of two vector spaces $L: V \times W \to \mathcal{U}$ to maps $L: V_1 \times V_2 \times \cdots \times V_m \to \mathcal{U}$ linear in each V_i separately. Then get the general tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ as the unique vector space such that any multilinear L induces a unique linear map L^* making diagram

$$\begin{tikzcd}
V_1 \times \cdots \times V_m \arrow[r, "L"] \arrow[d, "\pi"'] & \mathcal{U} \\
V_1 \otimes \cdots \otimes V_m \arrow[ur, "L^*"'] &
\end{tikzcd}$$

commute, where $\pi(v_1, \ldots, v_m) = v_1 \otimes \cdots \otimes v_m$.

Generalizing from the $m=2$ case, can get a basis for $V_1 \otimes \cdots \otimes V_m$ from bases of each V_i, $B_i = \{v_{i1}, \ldots, v_{id_i}\}$, $\dim(V_i) = d_i$, by taking all "basic tensors"

$$\mathcal{B} = \{v_{1j_1} \otimes v_{2j_2} \otimes \cdots \otimes v_{mj_m} \mid 1 \le j_i \le d_i\}$$

so $\dim(V_1 \otimes \cdots \otimes V_m) = d_1 d_2 \cdots d_m = \prod_{i=1}^{m} \dim(V_i)$.

This shows a clear difference from the direct sum, since

$$\dim(V_1 \oplus \cdots \oplus V_m) = d_1 + d_2 + \cdots + d_m.$$

$$= \dim\left(\prod_{i=1}^{m} V_i\right)$$ direct product

Applications: In the "tensor power" $V \otimes V \otimes \cdots \otimes V = V^{\otimes m}$ with m factors of the same vector space, V, we have an action of the symmetric (permutation) group S_m by permutation of the tensor factors: $\forall \sigma \in S_m$, $\sigma \cdot (v_1 \otimes \cdots \otimes v_m) = (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)})$.

Various interesting subspaces of $V^{\otimes m}$ can then be defined using this action.

Ex: Let $m = 2$, $S_2 = \{I, \sigma\}$, $\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\mathrm{Sym}^2(V) = \{v \in V \otimes V \mid \sigma \cdot v = v\} = \{\text{symm. tensors}\}$

$\mathrm{Anti}^2(V) = \{v \in V \otimes V \mid \sigma \cdot v = -v\}$
$= \{\text{anti-symm. tensors}\}$.

Let $B = \{v_1, \dots, v_n\}$ be a basis of V so
$\mathcal{B} = \{v_i \otimes v_j \mid 1 \le i, j \le n\}$ is a basis of $V \otimes V$.
$\forall v \in V \otimes V$, $v = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(v_i \otimes v_j)$ for some $a_{ij} \in F$. Let $A_v = [a_{ij}] \in F_n^n$ be the associated matrix determined by v.
Since $\sigma \cdot v = \sum_i \sum_j a_{ij}\, \sigma \cdot (v_i \otimes v_j) =$
$\sum_i \sum_j a_{ij}(v_j \otimes v_i) = \sum_j \sum_i a_{ji}(v_i \otimes v_j)$ we have

$\sigma \cdot v = v$ iff $a_{ij} = a_{ji}$ $\forall\, 1 \le i, j \le n$

iff $A_v = A_{\sigma \cdot v} = A_v^T$ iff A_v is symm.

and $\sigma \cdot v = -v$ iff $A_{\sigma \cdot v} = -A_v$ iff

$A_v^T = -A_v$ iff A_v is anti-symm.

So $Sym^2(V)$ has a basis $\{v_i \otimes v_j + v_j \otimes v_i \mid 1 \le i \le j \le n\}$ so $\dim(Sym^2(V)) = n(n+1)/2$

while $Anti^2(V)$ has basis $\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \le i < j \le n\}$ so $\dim(Anti^2(V)) = n(n-1)/2$

and $V \otimes V = Sym^2(V) \oplus Anti^2(V)$ is a dir. sum

dims: $n^2 = n(n+1)/2 + n(n-1)/2$