Let $A = [a_{ij}] \in \mathbb{C}_n^n$. For $1 \leq i \leq n$, let

$$R_i = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |a_{ij}|,$$

the sum of the absolute values of the non-diag. entries in $Row_i(A)$, and

$D_i = D(a_{ii}, R_i) \subseteq \mathbb{C}$ be the closed disk of radius R_i with center a_{ii}. So

$D_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i \}$.

Th. Every e-value of A is in some disk D_i.

Pf. Let $\lambda \in \mathbb{C}$ be an e-value of A. We can find an e-vector $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$ s.t. $AX = \lambda X$, and we may rescale X as follows. Among the coordinates $x_1, \ldots, x_n \in \mathbb{C}$, say x_i has maximum $|x_i|$. Then $\frac{1}{x_i} X$ has been rescaled so its i^{th} coordinate is 1 and all other coordinates $\frac{x_j}{x_i}$ have $\left|\frac{x_j}{x_i}\right| \leq 1$.

Call this rescaled e-vector $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$ where $x_i = 1$ and for $j \neq i$, $|x_j| \leq 1$.

From $AX = \lambda X$, looking at the i^{th} row, get

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i = \lambda, \quad \text{so} \sum_{j \neq i} a_{ij} x_j + a_{ii} = \lambda.$$

The triangle inequality then gives

$$|\lambda - a_{ii}| = \left|\sum_{j\neq i} a_{ij} x_j\right| \leq \sum_{j\neq i} |a_{ij}|\cdot|x_j| \leq \sum_{j\neq i} |a_{ij}| = R_i$$

which proves that $\lambda \in D_i$. □

<u>Cor.</u> The e-values of A are each in at least one of the Gershgorin disks defined from the columns of A, that is, for $1 \leq j \leq n$, let $C_j = \sum_{\substack{1\leq i\leq n \\ i\neq j}} |a_{ij}|$ be the sum of abs. values of the non-diag. entries in $\mathrm{Col}_j(A)$, and let

$$D^j = D(a_{jj}, C_j) = \{z \in \mathbb{C} \mid |z - a_{jj}| \leq C_j\}.$$

<u>Pf.</u> Apply Theorem to A^T (transpose of A). □