<u>Th</u>: If $L: V \to W$ is an invertible linear map then the function $L^{-1}: W \to V$ is also an invertible linear map, and $(L^{-1})^{-1} = L$.

<u>pf</u>. For any $w_1, w_2 \in W$, $\exists v_1, v_2 \in V$ s.t. $L(v_1) = w_1$ and $L(v_2) = w_2$ so $v_1 = L^{-1}(w_1)$ and $v_2 = L^{-1}(w_2)$. Since L is linear,

$L(v_1+v_2) = L(v_1) + L(v_2) = w_1 + w_2$ which means

$$v_1 + v_2 = L^{-1}(w_1 + w_2)$$ so we get

$L^{-1}(w_1 + w_2) = L^{-1}(w_1) + L^{-1}(w_2)$. For any $\alpha \in F$,

$L(\alpha v_1) = \alpha L(v_1) = \alpha w_1$ so $\alpha v_1 = L^{-1}(\alpha w_1)$ says $L^{-1}(\alpha w_1) = \alpha L^{-1}(w_1)$. Thus, L^{-1} is linear.

The fact that $L \circ L^{-1} = I_W$ and $L^{-1} \circ L = I_V$ means $L = (L^{-1})^{-1}$. $\square$

<u>Th</u>: If $L: F^n \to F^m$ is linear then $L = L_A$ for the matrix $A \in F^m_n$ such that $Col_j(A) = L(e_j)$ for $1 \le j \le n$, where $e_j = \begin{array}{l} \\ \\ \\ \text{row } j \\ \\ \end{array}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{row } j$.

<u>Pf</u>. For $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in F^n$ we have $X = \sum_{j=1}^{n} x_j e_j$ so $L(X) = \sum_{j=1}^{n} x_j L(e_j)$ by linearity of L. If $A \in F^m_n$ is the matrix with $Col_j(A) = L(e_j) \in F^m$ then

$$L_A(X) = AX = \sum_{j=1}^{n} x_j Col_j(A) = \sum_{j=1}^{n} x_j L(e_j) = L(X)$$

so $L_A = L$. $\square$

<u>Ex</u>: Let $L: F^2 \to F^2$ be $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. Then $L(e_1) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $L(e_2) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ so

$L\begin{bmatrix}x\\y\end{bmatrix} = xL\begin{bmatrix}1\\0\end{bmatrix} + yL\begin{bmatrix}0\\1\end{bmatrix} = x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$ and for $A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ $L_A(X) = AX = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0x-1y\\1x+0y\end{bmatrix}$ so $L\begin{bmatrix}x\\y\end{bmatrix} = L_A\begin{bmatrix}x\\y\end{bmatrix}$.

<u>Cor</u>: If $L_A : F^n \to F^n$ is invertible then $(L_A)^{-1} : F^n \to F^n$ is linear so $(L_A)^{-1} = L_B$ for some $B \in F^n_n$ and $B = A^{-1}$ so $(L_A)^{-1} = L_{(A^{-1})}$.

<u>Pf.</u> The Theorem on page 50 shows $(L_A)^{-1}$ is linear so the last Theorem on p.51 gives that $(L_A)^{-1} = L_B$ for the matrix B such that

$Col_j(B) = (L_A)^{-1}(e_j)$, $1 \le j \le n$. This gives

$L_A(\mathrm{Col}_j(B)) = e_j$ for $1 \le j \le n$ which means
$AB = I_n$ since $\mathrm{Col}_j(I_n) = e_j$. But $(L_A)^{-1} = L_B$
also means $L_B \circ L_A = I_{F^n}$ so $L_{BA} = L_{I_n}$ giving
$BA = I_n$ so $B = A^{-1}$. $\square$

This actually gives an algorithm to find A^{-1} (if it exists) by solving n linear systems, each with the coefficient matrix A; $AX = e_j$ gives $X = \mathrm{Col}_j(B) = \mathrm{Col}_j(A^{-1})$ with a different X for each $1 \le j \le n$. In terms of <u>row reduction</u> (to be discussed), solve all at one time:

$$[A \mid I_n] \xrightarrow[\text{reduce}]{\text{row}} [I_n \mid A^{-1}]$$ if A is invertible.

A more sophisticated viewpoint.

Def. For vector spaces V and W over F, let $Lin(V,W) = Hom(V,W) = \{L: V \to W \mid L \text{ is linear}\}$.

Define binary operation $+$ on $Lim(V,W)$ by:

$(L_1 + L_2)(v) = L_1(v) + L_2(v), \ \forall L_1, L_2 \in Lin(V,W), v \in V.$

Define scalar mult. on $Lin(V,W)$ by:

$(\alpha L)(v) = \alpha \cdot L(v), \ \forall L \in Lin(V,W), v \in V, \alpha \in F.$

Define the "zero lin. map" $O_W^V : V \to W$ by $O_W^V(v) = \theta_W, \ \forall v \in V$, (easily check it is linear).

Th: $Lin(V,W)$ with the above $+$ and $\cdot$, and zero vector O_W^V, is a vector space over F.

Pf. Exercise. $\square$

Ex: Let $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be lin. maps

$L_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+3y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ so $L_1 = L_A$ for $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$L_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ so $L_2 = L_B$ for $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Then

$$(L_1 + L_2)\begin{bmatrix} x \\ y \end{bmatrix} = L_1\begin{bmatrix} x \\ y \end{bmatrix} + L_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x+3y \end{bmatrix} + \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} x \\ 3x+3y \end{bmatrix}$$

$$L_{A+B}\begin{bmatrix} x \\ y \end{bmatrix} = (A+B)\begin{bmatrix} x \\ y \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Also, $(5 \cdot L_1)\begin{bmatrix} x \\ y \end{bmatrix} = 5 L_1\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x+y \\ 2x+3y \end{bmatrix} = \begin{bmatrix} 5x+5y \\ 10x+15y \end{bmatrix}$

$$= \begin{bmatrix} 5 & 5 \\ 10 & 15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = (5A)\begin{bmatrix} x \\ y \end{bmatrix} = L_{5A}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Looks like $L_A + L_B = L_{A+B}$ and $\alpha L_A = L_{\alpha A}$.

Th: $\mathrm{Lin}(F^n, F^m) = \{L : F^n \to F^m \mid L \text{ is linear}\}$
$= \{L_A : F^n \to F^m \mid A \in F_n^m\}$ and the map
$\mathcal{L} : F_n^m \longrightarrow \mathrm{Lin}(F^n, F^m)$ defined by $\mathcal{L}(A) = L_A$
is linear, bijective, invertible.

Pf. For $A, B \in F_n^m$, $\alpha \in F$, we have
$\mathcal{L}(A+B) = L_{A+B}$ and $\mathcal{L}(\alpha A) = L_{\alpha A}$. For any $X \in F^n$,
$L_{A+B}(X) = (A+B)X = AX + BX = L_A(X) + L_B(X) = (L_A + L_B)(X)$
$L_{\alpha A}(X) = (\alpha A)(X) = \alpha(AX) = \alpha(L_A(X)) = (\alpha L_A)(X)$
so $L_{A+B} = L_A + L_B$ and $L_{\alpha A} = \alpha L_A$. Thus, $\mathcal{L}$ is linear.
We have already seen that any linear $L : F^n \to F^m$
is $L = L_A$ for a unique $A \in F_n^m$, so $\mathcal{L}$ is <u>onto</u>.

We also know that if $L_A = L_B$ then $A = B$ so $\mathcal{L}(A) = \mathcal{L}(B)$ implies $A = B$ which means $\mathcal{L}$ is injective. Thus, $\mathcal{L}$ is bijective, so invertible. $\square$

Later we will see how to get an isomorphism between $Lin(V, W)$ and F^m_n when $\dim(V) = n$ and $\dim(W) = m$, but it will depend on the choice of a "basis" of V and a "basis" of W.

Here we implicitly used the "natural" or "standard" basis $\{e_1, \ldots, e_n\}$ of F^n and $\{e_1, \ldots, e_m\}$ of F^m. Apologies for using same notation in different spaces.