Math 507, Review of solving linear systems.

Let $A=[a_{ij}]\in F_n^m$, $B=[b_i]\in F^m$, $X=[x_j]\in F^n$.
The matrix equation $AX=B$ is equivalent to the system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad \text{for } 1\le i\le m.$$

We associate this with the $m\times(n+1)$ "augmented" matrix $[A|B]$ obtained from A by adding B as an extra column to A, separated by a vertical line to remind us that B was on the right side of the equation $AX=B$, and A was on the left side. Then n variables $x_1,\dots,x_n$ do not explicitly appear, but they are implicitly associated with the n columns of A. $[A|B]$ contains all the essential data of the linear system, and is like an accounting system where the columns are used to keep track of the data.

We "solve" a linear system by a multistep process, "*row reduction*", which changes the matrix $[A|B] \xrightarrow[\text{reduce}]{\text{row}} [C|D]$ by *elementary row operations* so that the lin. sys. $CX=D$ has the *same* solutions as $AX=B$. The goal is to **get** an *equivalent* $[C|D]$ whose solutions are "obvious", easily read off by a simple *interpretation* of $[C|D]$ in RREF.

There are 3 kinds of elem. row ops.

① Switch two rows,

② Multiply a row by a constant $0 \neq c \in F$,

③ Add a multiple of a row to another row.

These correspond to operations on the linear system which do not change the solutions.

<u>EX</u>. Linear system
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 4x_1 + 5x_2 &= 9 \end{aligned}$$
corresponds to
$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right]$$
whose solutions (if any) are "not obvious." Do the following elem. row ops.

$-4\,\text{Row}_1 + \text{Row}_2 \longrightarrow \text{Row}_2$, get
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -3 & -3 \end{array}\right]$$
$+(-4 \quad -8 \quad -12$

Multiply row 2 by $-\frac{1}{3}$:

$-\frac{1}{3}\text{Row}_2 \to \text{Row}_2$, get $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right]$.

Next: $-2\,Row_2 + Row_1 \to Row_1$, so

$$\begin{bmatrix} 1 & 2 & | & 3 \\ 0 & 1 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$0 \quad -2 \quad -2$

which "is" the linear system $X_1 = 1$, $X_2 = 1$

so the solution set is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ a single vector in $\mathbb{R}^2$ if $F = \mathbb{R}$ or in $\mathbb{Q}^2$ if $F = \mathbb{Q}$.

Of course, this simple example in $\mathbb{R}^2$ has a geometrical interpretation. Each equation has solutions forming a line in the plane $\mathbb{R}^2$ and their intersection is the single point $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ where both equations are satisfied:

$$1 + 2(1) = 3$$
$$4(1) + 5(1) = 9.$$

Does $AX = B$ always have exactly one solution for any choice of B? What can you say about $L_A : \mathbb{R}^2 \to \mathbb{R}^2$?

Ex: Linear system: $x_1 + 2x_2 + 3x_3 = 4$
$2x_1 + 5x_2 + 4x_3 = 7$

Solve by row reducing

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 5 & 4 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 7 & 6 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$-2 \; -4 \; -6 \; -8 \qquad 0 \; -2 \; 4 \; 2$

Interp.
$x_1 = -7r + 6$
$x_2 = 2r - 1$
$x_3 = r \in \mathbb{R}$ free variable

Solutions: $\left\{ \begin{bmatrix} -7r+6 \\ 2r-1 \\ r \end{bmatrix} = r \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; r \in \mathbb{R} \right\}$

expressed in terms of free variable (parameter) $x_3 = r \in \mathbb{R} = F$ in this example.

Note: Got infinitely many solutions, one $\forall r \in \mathbb{R}$.
They form a line in $\mathbb{R}^3$ passing through $\begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$ parallel to line $\left\langle \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \right\rangle$ through origin.
Intersection of 2 planes in $\mathbb{R}^3$ was a line.

Questions about last example:

For $A=\begin{bmatrix}1&2&3\\2&5&4\end{bmatrix}$, is linear map $L_A:\mathbb{R}^3\to\mathbb{R}^2$ onto? injective? What is $\ker(L_A)$? $\mathrm{Im}(L_A)$?

Answers: Onto: Does $AX=B$ have a solution $\forall B\in\mathbb{R}^2$

$$\left[\begin{array}{ccc|c}1&2&3&b_1\\2&5&4&b_2\end{array}\right]\to\left[\begin{array}{ccc|c}1&2&3&b_1\\0&1&-2&b_2-4b_1\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&7&9b_1-2b_2\\0&1&-2&b_2-4b_1\end{array}\right]$$

$-2\ -4\ -6\ -4b_1 \qquad 0\ -2\ 4\ \ 8b_1-2b_2$

Interp: $x_1=-7r+9b_1-2b_2$
$x_2=2r-4b_1+b_2$
$x_3=r\in\mathbb{R}$ free

Yes, can find ∞-many solutions $\forall B\in\mathbb{R}^2$. Yes, L_A is onto.

Inj? No, ∞-many X hit same B.

$$\ker(L_A)=\left\{r\begin{bmatrix}-7\\2\\1\end{bmatrix}\in\mathbb{R}^3\ \middle|\ r\in\mathbb{R}\right\}=\left\langle\begin{bmatrix}-7\\2\\1\end{bmatrix}\right\rangle$$

$\mathrm{Im}(L_A)=\mathbb{R}^2$ since L_A is onto.