Ex: Is $A=\begin{bmatrix}1&1&-1\\2&3&-5\\1&3&-7\end{bmatrix}$ invertible? Try to find inverse by row reducing

$$[A|I_3]=\left[\begin{array}{ccc|ccc}1&1&-1&1&0&0\\2&3&-5&0&1&0\\1&3&-7&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&1&-1&1&0&0\\0&1&-3&-2&1&0\\0&2&-6&-1&0&1\end{array}\right]\to$$

+ +

$-2\ -2\ 2\ -2\ 0\ 0$ $\qquad$ $0\ -2\ 6\ 4\ -2\ 0$

$-1\ -1\ 1\ -1\ 0\ 0$

$$\left[\begin{array}{ccc|ccc}1&1&-1&1&0&0\\0&1&-3&-2&1&0\\0&0&0&3&-2&1\end{array}\right]$$

But the row of zeros on the last row of left side stands for the equations $0x_1+0x_2+0x_3=3$ or -2 or 1 in the 3 lin. systems needing to be solved. $0\neq 3$ so system is <u>not</u> consistent, no solutions, A is <u>not</u> invertible.

What can you say about $L_A:\mathbb{R}^3\to\mathbb{R}^3$ for this A?

Exercise: Is $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -5 \\ 1 & 3 & -6 \end{bmatrix}$ invertible? If so, find A^{-1} and verify that $AA^{-1} = I_3 = A^{-1}A$ for your answer. L78

Def. Say $A = [a_{ij}] \in F^m_n$ is in Reduced Row Echelon Form (RREF) when:

1. All zero rows are at the bottom.
2. The left-most non-zero entry of each non-zero row is a $1 \in F$. Call those entries "leading 1's".
3. If $a_{ij} = 1$ and $a_{kl} = 1$ are leading 1's and $i < k$ then $j < l$.
4. If $a_{ij} = 1$ is a leading 1 then all other entries in column j are 0.

Examples: $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **is** in RREF.

$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **is not** in RREF but $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **is** in RREF.

Def. On $\mathbb{F}^m_n$ define relation $\underset{row}{\sim}$ (row equival.) by $A \underset{row}{\sim} B$ when there is a finite sequence of elem. row ops. which changes A into B.

Th. Each type of elem. row op. is invertible, and its inverse is an elem. row op. of the same type.

Pf. A switcher is its own inverse.
Inverse of a multiplier by $0 \neq c \in F$ is the multiplier by $c^{-1} \in F$.
Inverse of adder $c\,Row_i + Row_j \to Row_j$ is adder $-c\,Row_i + Row_j \to Row_j$. □

Th: The relation $\underset{row}{\sim}$ on F^m_n is an __equivalence relation__, that is, reflexive, symmetric and transitive. So F^m_n has a partition into disjoint __equivalence classes__ (mutually row equiv. matrices) 80

$\overline{A} = \{M \in F^m_n \mid M \underset{row}{\sim} A\}$ and $\overline{A} = \overline{B}$ iff $A \underset{row}{\sim} B$.

Def: Suppose $A \in F^m_n$ and $A \underset{row}{\sim} B$ with B in RREF. The __pivot__ columns of A are the columns where B has leading 1's. The number of leading 1's in B is called the __rank of A__, denoted $\text{rank}(A)$, and it equals the number of non-zero rows of B.

__Note:__ $\text{rank}(A) \le \text{Min}(m,n)$ since can't have more leading 1's than rows (columns) to put them in.

Suppose $\underset{m\times n}{[A} | \underset{m\times 1}{B]} \xrightarrow{r.r.} [C|D]$ with C in RREF and the pivot columns are columns $j_1,\dots,j_r$ with $1 \le j_1 < j_2 < \dots < j_r \le n$, $r = \mathrm{rank}(A)$. Then we can immediately write down the solutions to the lin. sys., if it is consistent, as follows.

① If C has any zero rows but the number in column D is not zero in such a row, then the system is not consistent.

② For each variable x_j with $j \notin \{j_1,\dots,j_r\}$ (a non-pivot variable), $x_j \in F$ is free (parameter) allowed to have any value in field F.

③ For each <u>pivot</u> variable x_{j_k}, $1 \le k \le r$, the row in $[C|D]$ containing that leading 1, is an equation of the form

$1x_{j_k} + \sum c_{i_k j} x_j = d_{i_k}$ where the row containing this leading 1 is numbered i_k and the sum is only over <u>free</u> variables x_j for $j > j_k$. This gives the formula

$x_{j_k} = -\sum c_{i_k j} x_j + d_{i_k}$ for all <u>pivot</u> variables in terms of <u>free</u> variables and the constants in D.

<u>Note</u>. With a total of n variables and r pivot variables, there are exactly $n-r$ free variables.

Example: Suppose $[A|B] \xrightarrow{\text{r.r.}} [C|D] =$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & -2 & 5 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r = 2$
$j_1 = 1,\ j_2 = 3$
$i_1 = 1,\ i_2 = 2$

Interp (say $F = \mathbb{R}$)

$x_1 = -3r - 2s - 4t + 7$
$x_2 = r \in \mathbb{R}$ free
$x_3 = 2s - 5t + 9$
$x_4 = s \in \mathbb{R}$ free
$x_5 = t \in \mathbb{R}$ free

So solution set is

$$\left\{ \begin{bmatrix} -3r-2s-4t+7 \\ r \\ 2s-5t+9 \\ s \\ t \end{bmatrix} \in \mathbb{R}^5 \;\middle|\; r, s, t \in \mathbb{R} \right\}$$

$$= \left\langle \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\rangle + \begin{bmatrix} 7 \\ 0 \\ 9 \\ 0 \\ 0 \end{bmatrix}$$

is a translated subspace, called an <u>affine space</u>.

Example: Suppose $A \in F^n_n$ is square and $A \underset{\text{row}}{\sim} I_n$. Then $\underset{n\times n}{[A} | \underset{n\times 1}{B]} \xrightarrow{\text{r.r.}} [I_n | D]$ means there is a unique solution since the interp. of $[I_n | D]$ as equations is

$$\begin{aligned} x_1 &= d_1 \\ x_2 &= d_2 \\ &\vdots \\ x_n &= d_n \end{aligned}$$

so get $\left\{ \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \right\}$ as solution set, a single point in F^n.

Th. Let $A \in F^m_n$ with $r = \text{rank}(A)$.

(a) $L_A : F^n \to F^m$ is <u>injective</u> iff $r = n$.

(b) L_A is <u>surjective</u> iff $r = m$.

(c) L_A is bijective iff $m = r = n$.

Pf. (a) We know L_A is inj. iff $Ker(L_A) = \{0^n_1\}$ 185
But $Ker(L_A) = \{X \in F^n \mid AX = 0^m_1\}$ is found by
$[A \mid 0^m_1] \xrightarrow{r.r.} [C \mid 0^m_1]$. If $r < n$ there are
$n - r > 0$ free variables giving non-zero solutions.
Also $r \le Min(m,n)$ so $r \le n$. Only $r = n$
gives $Ker(L_A) = \{0^n_1\}$.

(b) L_A is surj. iff $\forall B \in F^m_1$ $AX = B$ is
consistent. Solve it by row reducing
$[A \mid B] \xrightarrow{r.r.} [C \mid D]$ with C in RREF. $r \le m$.
If $r < m$ C has at least one zero row
(actually has $m - r$ zero rows) so can get

inconsistency (no solutions) if entry on that row in D is $\neq 0$. Only way to always be consistent $\forall B$ is if $r = m$.

<u>Note</u>: Can get <u>consistency</u> <u>conditions</u> on $B \in \text{Range}(L_A)$ by using letters $B = [b_i]$ and then get formulas for entries of D in terms of $b_1, \ldots, b_m$, so each zero row of C gives a condition on these b_i.

(c) Follows immediately from (a) and (b).

<u>Cor</u>: For $A \in F^m_n$, $L_A : F^n \to F^m$ cannot be invertible if $m \neq n$.