Def. Matrix $E \in F^n_n$ obtained from I_n by an elem. row op. done to I_n is called the *elem. matrix* associated with that elem. row op.

Examples: ① $E_s = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the elem. matrix associated with the switcher row op. $R_1 \longleftrightarrow R_3$.

② $E_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the elem. matrix associated with the multiplier row op. $cR_2 \longrightarrow R_2$.

③ $E_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$ is the elem. matrix associated with the adder row op. $cR_1 + R_3 \longrightarrow R_3$.

Note: $E_S^{-1} = E_S$, $E_M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$.

<u>Th</u>: Let $E \in F^m_m$ be the elem. matrix associated with an elem. row op. Then $\forall A \in F^m_n$, $B = EA$ is the matrix obtained from A by doing that row op. to A, so $EA \underset{\text{row}}{\sim} A$.

<u>Ex</u>: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4c & 5c & 6c \end{bmatrix}$

$\underset{E = E_{Adder}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}}$. Continuing to row reduce B we have

$E_M B = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = C$ and then

$E'_{Adder} C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = D$ is in RREF.

So we found a finite sequence of elem. matrices, $E_1 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

such that $E_3 E_2 E_1 A = D$ in RREF. The row reduction of A to D was achieved by the left multiplication of elem. matrices corresponding to the elem. row ops.

<u>Th.</u> For $A, B \in F^m_n$, $A \underset{row}{\sim} B$ iff $B = E_t \cdots E_2 E_1 A$ for some elem. matrices $E_1, \ldots, E_t \in F^m_m$.

<u>Th.</u> For $A \in F^n_n$, A is invertible iff $A \underset{row}{\sim} I_n$ iff A is a product of elem. matrices.

Pf. We have seen that A is invertible iff $[A|I_n] \xrightarrow{\text{r.r.}} [I_n|B]$ so that $AB = I_n$ and $A \underset{\text{row}}{\sim} I_n$. The row reduction process on both sides of $[A|I_n]$ can be achieved using elem. matrices, $E_1, \dots, E_t$,

$$[A|I_n] \to [E_1A|E_1I_n] \to [E_2E_1A|E_2E_1I_n] \to \cdots$$

so finally $[E_t \cdots E_1A|E_t \cdots E_1I_n] = [I_n|B = A^{-1}]$.

From the right side get $E_t \cdots E_1 = B = A^{-1}$ and from the left get $(E_t \cdots E_1)A = I_n$.

Each elem. matrix is invertible and its inverse is also elem. (of same type), so

$A = E_1^{-1}E_2^{-1} \cdots E_t^{-1}$ is a product of elem. matrices. $\square$

<u>Cor</u>. For $A, B \in F^m_n$, $A \underset{\text{row}}{\sim} B$ iff $B = QA$ for some invertible $Q \in F^m_m$.

<u>Th</u>: Let $E \in F^n_n$ be the elem. matrix obtained from I_n by doing an elem. <u>col</u>. op to I_n. Then $\forall A \in F^m_n$, $B = AE$ is the matrix obtained from A by doing that col. op. to A.

<u>Ex</u>: $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 6 & 3 \end{bmatrix}$,

$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 5 \\ 9 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \\ 3 & -6 \end{bmatrix}$

<u>Note</u>: Col. ops. do <u>not</u> help to solve lin. systems!

Def: For $A, B \in F^m_n$, say $A \underset{col}{\sim} B$ (column equivalent) when B can be obtained from A by a finite sequence of elem. col. ops.

Th: $\underset{col}{\sim}$ is an equiv. relation on F^m_n (symm, reflex, trans.) and $A \underset{col}{\sim} B$ iff $B = AP$ for some invertible $P \in F^n_n$ (which is a product of elem. matrices).

Th. Each $A \in F^m_n$ is $\underset{col}{\sim}$ to a unique matrix in Reduced Column Echelon Form (RCEF).

Def. For $A, B \in F^m_n$ say $A \underset{row/col}{\sim} B$ (row/col equivalent) when $B = QAP$ for some invertible matrices $P \in F^n_n$ and $Q \in F^m_m$.

What is the "nicest" B s.t. $A \underset{row/col}{\sim} B$?

Basic Concepts for Vector spaces:

<u>Def.</u> Let $S \subseteq V$ be any subset of v.s. V over field F. Say S is <u>independent</u> when

$\sum_{i=1}^{m} x_i v_i = \theta$ for $v_1, \dots, v_m \in S$, $x_1, \dots, x_m \in F$

implies $x_i = 0$ for $1 \le i \le m$. Otherwise, say S is <u>dependent</u>, and call the above equation (with at least one non-zero coeff.) a <u>dependence relation on S</u>.

The empty set is considered to be <u>indep.</u>

If S is an infinite set, indep. means any choice of finitely many $v_i \in S$ satisfies the condition above. For S finite, write $S = \{v_1, \dots, v_m\}$.

Ex. $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is indep. since

$x_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0$. But

$T = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is **dep.** since $1 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Th. In any v.s. V, $S = \{v\}$, where $v \neq \theta$, is indep but $T = \{\theta_V\}$ is **dep.** So $\{v\}$ is indep. iff $v \neq \theta_V$.

Pf. Recall that $\alpha \cdot v = \theta_V$ implies $\alpha = 0 \in F$ or $v = \theta_V \in V$. So given $\alpha \cdot v = \theta_V$ we get: $v \neq \theta_V \Rightarrow \alpha = 0$ and

$\alpha \neq 0 \Rightarrow v = \theta_V$. $\square$

<u>Ex</u>: $V=\mathbb{R}^2$, $S=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is indep. since

$$x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_1=x_2=0.$$

But $T=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\}$ is <u>dep</u> since

$-2\begin{bmatrix}1\\0\end{bmatrix}-3\begin{bmatrix}0\\1\end{bmatrix}+1\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, that is,

$x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}+x_3\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ has non-trivial solution(s).

Corresp. matrix calculation: Does lin sys.

$\underbrace{\left[\begin{array}{ccc|c}1&0&2&0\\0&1&3&0\end{array}\right]}_{A}$ have any non-triv. solutions?

Yes, since $\operatorname{rank}(A)=2<n=3$

so get $n-r=3-2=1$ free var.

Th. In $V = F^m$ let $S = \{v_1, \ldots, v_n\} \subseteq V$ with $n > m$. Then S is <u>dep.</u>

<u>Pf.</u> Look at the lin. sys. corresponding to the vector equation: $\sum_{j=1}^{n} x_j \cdot v_j = 0_1^m$.

Write $v_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \in F_1^m$, $A = [a_{ij}] \in F_n^m$ with $X \in F^n$. So the lin. sys. is $\underset{(m \times n)(n \times 1)}{AX} = 0_1^m$.

Does $[A \mid 0_1^m]$ have any free variables in its solution set?

For $n > m$, $r = \text{rank}(A) \le \text{Min}(m, n) = m < n$, so must have $n - r > 0$ free variables, so S is dep. $\square$

Note: Each free variable gives a dep. relation on S. Examples later.

Cor. In F^m, the maximum size of an indep. set is m.

EX: The set $S = \{e_1, \dots, e_m\}$ of "standard basis" vectors in F^m is indep. since

$$\sum_{i=1}^{m} x_i e_i = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ iff each $x_i = 0$.

EX: In F^m_n, the set $S = \{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is indep since $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij} = A = [a_{ij}] = 0^m_n$ iff each $a_{ij} = 0$.

This set is called the "standard basis" of F^m_n.