Th: Let $T \subseteq S \subseteq V$ be subsets of v.s. V /98

(a) If S is <u>indep.</u> then T is <u>indep.</u>

(b) If T is <u>dep.</u> then S is <u>dep.</u>

<u>Pf.</u> (a) Suppose S is indep. Any lin. comb. of vectors in T is also a lin. comb. of vectors in S since $T \subseteq S$. So if the combo. is θ_V, all its coeffs are 0 by indep. of S, so T is indep.

(b) Is the contrapositive of (a). □

(a) Subset of indep set is indep

(b) If a set is dep, so is any containing "superset".

<u>Ex</u>: In $V = \mathbb{R}^2_2$, is $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \right\}$ 99

indep or dep? Solve

$$\sum_{j=1}^{4} x_j v_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

4 eq's in 4 var's:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{r.r.}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

has <u>only</u> the trivial solution, $x_1 = x_2 = x_3 = x_4 = 0$.

So S is <u>indep</u>. So any subset of S is indep.

<u>Ex</u>: Is $T = \left\{ \underset{w_1}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}, \underset{w_3}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{w_4}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \right\}$ indep or dep?

$$\sum_{j=1}^{4} x_j w_j = \begin{bmatrix}0&0\\0&0\end{bmatrix} = x_1\begin{bmatrix}1&1\\0&0\end{bmatrix} + x_2\begin{bmatrix}0&0\\1&1\end{bmatrix} + x_3\begin{bmatrix}1&0\\1&0\end{bmatrix} + x_4\begin{bmatrix}0&1\\0&1\end{bmatrix}$$ $\underline{100}$

is lin. sys.

$$\begin{aligned} x_1 \quad\; + x_3 \quad\;\; &= 0 \\ x_1 \quad\quad\quad\; + x_4 &= 0 \\ x_2 + x_3 \quad\;\; &= 0 \\ x_2 \quad\quad\quad\; + x_4 &= 0 \end{aligned}$$

So we need to solve by row red.

$$\left[\begin{array}{cccc|c} 1&0&1&0&0\\ 1&0&0&1&0\\ 0&1&1&0&0\\ 0&1&0&1&0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1&0&0&1&0\\ 0&1&0&1&0\\ 0&0&1&-1&0\\ 0&0&0&0&0 \end{array}\right] \quad \begin{aligned} x_1 &= -r \\ x_2 &= -r \\ x_3 &= r \\ x_4 &= r \in \mathbb{R} \text{ free} \end{aligned}$$

Get nontrivial solutions so T is <u>dep</u>.

$r=1$ gives the dep. relation

$$-w_1 - w_2 + w_3 + w_4 = 0_2^2$$

so

$$w_4 = w_1 + w_2 - w_3$$

Recall the definition of the span of S for $S \subseteq V$ is $\langle S \rangle = \left\{ \sum_{i=1}^{m} x_i v_i \in V \,\middle|\, v_i \in S, x_i \in F \right\}$

$= \{$finite lin. combinations from $S\}$.

__Def__. Say $S \subseteq V$ __spans__ V when $\langle S \rangle = V$.

For subspace $W \leq V$, S spans W means $\langle S \rangle = W$. Of course, $\forall v_i \in S$, $\exists v_i \in \langle S \rangle$.

__Ex__. The "std. basis" $S = \{e_1, \dots, e_n\} \subseteq F^n$ spans F^n since $\langle S \rangle = \left\{ \sum_{j=1}^{n} a_j e_j = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n \,\middle|\, a_j \in F \right\} = F^n$.

but any proper subset of S does not span F^n, only give a proper subspace of F^n.

Given $S \subseteq V$, to find $\langle S \rangle$, set up an $\underline{L102}$ equation $\sum_{j=1}^{n} x_j \cdot v_j = v$ for a general $v \in V$ and $S = \{v_1, \dots, v_n\}$. Convert to a lin. sys.

$$\underset{\text{"as columns"}}{\left[\text{"}S\text{"} \,\middle|\, \text{"}v\text{"}\right]} = [A|B] \xrightarrow{\text{r.r.}} \underset{\text{RREF}}{[C|D]}$$

$\underline{\text{Interpret}}$ zero rows of C give consistency conditions which tell which $v \in V$ are in $\langle S \rangle$. If $\operatorname{rank}(A) = m$, get no zero rows in C, no consistency conditions so $\langle S \rangle = V$. If $\operatorname{rank}(A) < m$, do get conditions allowing precise description of $\langle S \rangle = \{v \in V \mid \text{consistency conditions hold}\}$.

Def. Say $S \subseteq V$ is a basis of V when S is indep and S spans V. This applies to $W \leq V$, as well, so we say $T \subseteq V$ is a basis of W when T is indep. and $\langle T \rangle = W$.

Ex: The "standard basis" of F_n^m given before is a basis of F_n^m and it consists of $m \cdot n$ vectors. This includes the cases of F^m with std. basis $S = \{e_1, \ldots, e_m\}$ and of F_n with std. basis $S = \{e_1, \ldots, e_n\}$ row vectors.

Th: If S and T are any two bases of V, each with a finite number of vectors, say $S = \{v_1, \ldots, v_m\}$ and $T = \{w_1, \ldots, w_n\}$, then $m = n$.

Def. The number of vectors in any basis of V is called the **dimension** of V, denoted $\dim(V)$.

Ex: $\dim(F^m) = m$, $\dim(F_n) = n$,
$\dim(F_n^m) = m \cdot n$.

If there are infinitely many vectors in a basis of V, say V is infinite dim'l, write $\dim(V) = \infty$.

Ex: $V = F[t] = \langle 1 = t^0, t^1, t^2, \dots \rangle = \langle S \rangle$ for basis $S = \{1, t, t^2, \dots\} = \{t^i \mid 0 \le i \in \mathbb{Z}\}$ "std. basis"

so $\dim(F[t]) = \infty$.

Ex: Let $W = \{p(t) \in F[t] \mid \deg(p) \leq 3,\ p'(1)=0\}$ 105

① Check that $W \leq F[t]$.

② Get a precise description of all $p(t) \in W$ which allows you to find a basis for W.

Solution: $\deg(p) \leq 3$ means $p(t) = \sum_{i=0}^{3} c_i t^i$.

$$p'(t) = \sum_{i=0}^{3} i c_i t^{i-1} = c_1 + 2c_2 t + 3c_3 t^2,$$ so

$0 = p'(1) = c_1 + 2c_2 + 3c_3$ is the condition on coefficients, no restriction on $c_0 \in F$.

Lin. sys.: 4 variables; c_0, c_1, c_2, c_3, one equation

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

Let $c_0 = a$, $c_2 = b$, $c_3 = c$ free

Interp.

$c_0 \in F$ free

$c_1 = -2c_2 - 3c_3$

$c_2 \in F$ free

$c_3 \in F$ free

$$W = \{a1 + (-2b-3c)t + bt^2 + ct^3 = a1 + b(-2t+t^2) + c(-3t+t^3) \mid a, b, c \in F\} = \langle 1, t^2-2t, t^3-3t \rangle$$

Ex. Find a basis for $W = \{X \in \mathbb{R}^3 \mid AX = 0^2_1\}$ if

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$. Solution: We already solved lin. sys.

$[A \mid 0^2_1] \xrightarrow{\text{r.r.}} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ on page 75, got $\begin{aligned} x_1 &= -7r \\ x_2 &= 2r \\ x_3 &= r \in \mathbb{R} \text{ free} \end{aligned}$

so $W = \left\{ r\begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, r \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \right\rangle$

so $\left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \right\}$ spans W and is indep, so it is a basis for W. $\dim(W) = 1$. $W = \mathrm{Ker}(L_A)$.

Find a basis for $\mathrm{Range}(L_A) = \{AX \in \mathbb{R}^2 \mid X \in \mathbb{R}^3\}$

$= \left\{ \sum_{j=1}^{3} x_j \,\mathrm{Col}_j(A) = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}$

$= \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\rangle$ Columns of A span $\mathrm{Range}(L_A)$

but the set of those 3 vectors in $\mathbb{R}^2$ is dep. since $3 > 2$. How do we get a basis?

Method 1: Find dep. rel. on set of spanning vectors, use to remove "redundant" vectors, cut down spanning set to an indep. spanning set.

From $Ker(L_A)$ basis vector, get dep. rel.

$$-7\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}2\\5\end{bmatrix}+1\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad \text{so}$$

$$\begin{bmatrix}3\\4\end{bmatrix}=7\begin{bmatrix}1\\2\end{bmatrix}-2\begin{bmatrix}2\\5\end{bmatrix}$$

which allows any lin. comb.

$$x_1\begin{bmatrix}1\\2\end{bmatrix}+x_2\begin{bmatrix}2\\5\end{bmatrix}+x_3\begin{bmatrix}3\\4\end{bmatrix}=x_1\begin{bmatrix}1\\2\end{bmatrix}+x_2\begin{bmatrix}2\\5\end{bmatrix}+x_3\left(7\begin{bmatrix}1\\2\end{bmatrix}-2\begin{bmatrix}2\\5\end{bmatrix}\right)$$

$$=(x_1+7x_3)\begin{bmatrix}1\\2\end{bmatrix}+(x_2-2x_3)\begin{bmatrix}2\\5\end{bmatrix}\in\Big\langle \underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{Col_1(A)}, \underbrace{\begin{bmatrix}2\\5\end{bmatrix}}_{Col_2(A)}\Big\rangle$$

so $Col_3(A)=\begin{bmatrix}3\\4\end{bmatrix}$ was redundant in the spanning set for $Range(L_A)$.

But $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}\right\}$ is indep since $\left[\begin{array}{cc|c}1&2&0\\2&5&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$ has only triv. sol'n. Thus, a basis for Range(L_A) is $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}\right\}$, the pivot columns of A.

__Th.__ Let $A \in F^m_n$, $L_A : F^n \to F^m$ for $L_A(X) = AX$.

(a) $Ker(L_A) = \{X \in F^n \mid AX = 0^m_1\}$ is found by row reducing $[A | 0^m_1] \xrightarrow{r.r.} [C | 0^m_1]$ if $rank(A) = r$. Let the solutions be written as

__Interp__

$$\left.\begin{array}{l} x_1 = \cdots \\ x_2 = \cdots \\ \vdots \\ x_n = \cdots \end{array}\right\} \text{formulas in terms of } n-r \text{ free var's}$$

$X = f_1 K_1 + \cdots + f_{n-r} K_{n-r}$ for free variables $f_1, \ldots, f_{n-r}$ and constant vectors $K_1, \ldots, K_{n-r} \in F^n$.

Then a basis for $Ker(L_A)$ is $\{K_1, \ldots, K_{n-r}\}$.

(b) $\mathrm{Range}(L_A) = \{AX \in F^m \mid X \in F^n\}$

$= \left\{ \sum_{j=1}^{n} x_j \, \mathrm{Col}_j(A) \in F^m \,\middle|\, x_j \in F \right\} = \langle \mathrm{Col}_1(A), \ldots, \mathrm{Col}_n(A) \rangle$

is the span of the columns of A, called the column space of A, denoted by $\mathrm{Col}(A)$.

A basis for $\mathrm{Col}(A)$ consists of just the pivot columns of A, that is, the r columns in $\{\mathrm{Col}_j(A) \mid x_j \text{ is not a free variable in } \mathrm{Ker}(L_A)\}$.

<u>Cor:</u> $\underbrace{\dim(\mathrm{Ker}(L_A))}_{= n-r} + \underbrace{\dim(\mathrm{Range}(L_A))}_{= r} = n$