Tensor Product

Ezekiel Lemann

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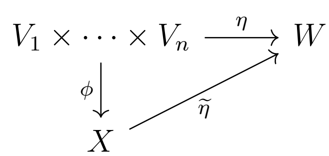
Problem 1: Suppose we have an extension of fields $F \to K$ and V is F-vector space. We may want to modify V in such a way that the modified "V" becomes a K-vector space. For a familiar example, take $F = \mathbb{R}$, $K = \mathbb{C}$, and $V = \mathbb{R}^n, M_n(\mathbb{R}), \mathbb{R}[t]$ or some other real vector space.

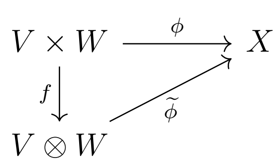
Problem 2: Suppose that V_1, V_2, W are K-vector spaces and that we have a bilinear map $\eta : V_1 \times V_2 \to W$ which we would like to learn about. The more general situation is where V_i are K-vector spaces and $\eta : V_1 \times V_2 \times \cdots \times V_n \to W$ is a multilinear map.

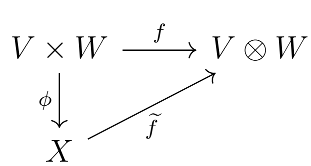
To work on these problems we would like to construct a vector space X together with a multilinear map $\phi : V_1 \times \cdots \times V_n \to X$ such that given any multilinear $\eta : V_1 \times \cdots \times V_n \to W$ there exists a unique homomorphism $\widetilde{\eta}$ such that the following diagram of K-vector spaces commutes.



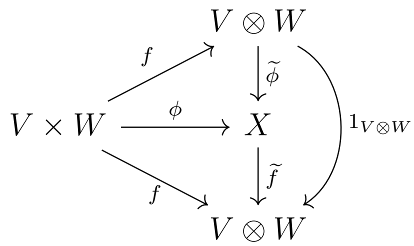
Notation: $X = V_1 \otimes \cdots \otimes V_n = \bigotimes_{i=1}^n V_i$. We can specify what field we are tensoring over by subscript $\otimes_F$.

Uniqueness: Let V, W be K-vector spaces and let $(X, \phi), (V \otimes_K W, f)$ both satisfy the property of the tensor product. Then we have the following commutative diagrams:

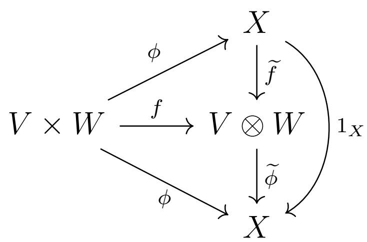




Which means that $\phi = \widetilde{\phi} f, f = \widetilde{f}\phi$. Now we examine the larger diagram:



We have that $f = \tilde{f}\tilde{\phi} f$ which implies $1_{V \otimes W} = \tilde{f}\tilde{\phi}$. By the same argument the diagram:



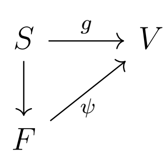
shows that $1_X = \tilde{\phi}\tilde{f}$. Therefore the tensor product $V \otimes_F W$, if it exists, is unique up to unique isomorphism.

Note: The isomorphism is unique since the induced map is assumed to be unique. If $(X, f), (Y, g)$ are tensor products of V, W, two K-vector spaces, then it is natural to consider an *isomorphism of tensor products* as isomophism $H : X \to Y$ such that $H \circ f = g$. This means that X is identified with Y in such a way that preserves universal property of tensor products. Of course there are many isomorphisms between the K-vector spaces X and Y, but the isomorphism given above is between (X, f) and (Y, g).

Existence:

Given a set S and a field K the *free K-vector space on S* is $\bigoplus_{s \in S} K$. For $|S| = \infty$ we have $\bigoplus_{s\in S} K := \{x \in \prod_{s\in S} K : x_i = 0$ for all but a finite number of indices $\}$. If $x \in \bigoplus_{s\in S} K$ then $x = k_1 s_1 + \cdots + k_n s_n$ for $k_i \in K, s_i \in S$.

Exercise: The free K-vector space F on the set S has the following special property: Given any K-vector space V and function $g : S \to V$ there is a unique homomorphism ψ making the diagram commute:



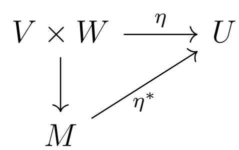
Construction: To construct the tensor product $V \otimes_K W$ take the free vector space on the set $V \times W$, that is $M = \bigoplus_{(v,w) \in V \times W} K$. Now we define $N \leq M$ to be the subspace generated by all elements of the form

$$(v + v', w) - (v, w) - (v', w), (\alpha v, w) - \alpha(v, w)$$
$$(v, w + w') - (v, w) - (v, w'), (v, \alpha w) - \alpha(v, w).$$

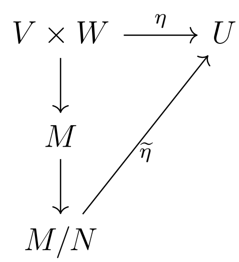
Now define $V \otimes_K W := M/N$. The notation $v \otimes w := (v, w) + N$ is used. The map $f : V \times W \to V \otimes_K W$ is given by $(v, w) \mapsto v \otimes w$. We have the following properties:

$$v \otimes (w + w') = v \otimes w + v \otimes w'$$
$$(v + v') \otimes w = v \otimes w + v' \otimes w$$
$$\alpha v \otimes w = v \otimes \alpha w = \alpha(v \otimes w)$$

These properties imply that f is bilinear.
Now suppose that $\eta : V \times W \to U$ is K-bilinear. Considering η as a set map gives us a unique homomorphism $\eta^* : M \to U$ (by the exercise) making the diagram commute:



Since we supposed that η is bilinear we have that $\eta^*(N) = 0$, that is $N \leq \ker(\eta^*)$. Therefore there exists a unique map $\widetilde{\eta}$ such that the diagram:



is commutative. Therefore $M/N = V \otimes_K W$ satisfies the the property of the tensor product.

What is $\dim(V \otimes_K W)$? Given bases for V and W what is a basis for $V \otimes_K W$?

Prop.1 Given K-vector spaces V, W, U there is a unique isomorphism

$$(V \otimes W) \otimes U \to V \otimes (W \otimes U)$$

given by $(v \otimes w) \otimes u \mapsto v \otimes (w \otimes u)$.

Prop.2 Given K-vector spaces V, W there is a unique isomorphism

$$V \otimes W \to W \otimes V$$

given by $v \otimes w \mapsto w \otimes v$.
Note: The isomorphism is unique in the sense explained after the proof of uniqueness of tensor product. Vector space isomorphisms such as $L : V \otimes W \to W \otimes V$ defined by $L(v \otimes w) = \alpha w \otimes v$ for $1 \neq \alpha \in K$ do not preserve commutativity. Given $(V \otimes W, \phi), (W \otimes V, \psi)$ we have that $L \circ \phi(v, w) = L(v \otimes w) = \alpha w \otimes v \neq w \otimes v = \psi(w, v)$. Put another way, $(W \otimes V, L \circ \phi)$ is not a tensor product.

Prop.3 Let $V = \bigoplus_{i=1}^n V_i$ and W be K-vector spaces. There is an isomorphism

$$f : W \otimes_K V \to \bigoplus_{i=1}^n (W \otimes_K V_i).$$

given by $w \otimes (v_i) \mapsto (w \otimes v_i)$
Notation: $(v_i) = (v_1, \dots, v_n)$

Sketch: Define the map $g : W \times V \to \bigoplus_{i=1}^n (W \otimes_K V_i)$ by $(w, (v_i)) \mapsto (w \otimes v_i)$. This map is bilinear. Partial check:

$$g(w + w', (v_i)) = (w + w') \otimes v_i = (w \otimes v_i) + (w' \otimes v_i)$$
$$g(w, (v_i)) + g(w', (v_i)) = (w \otimes v_i) + (w' \otimes v_i)$$

Then by the property of the tensor product we have a homomorphism

$$f : W \otimes_K V \to \bigoplus_{i=1}^n (W \otimes_K V_i)$$

with $f(w \otimes (v_i)) = w \otimes v_i$. Now we can construct an inverse for f. Let $\phi_j : V_j \to V$ denote the inclusion. For $v_j \in V_j$we have $\phi_j v_j = (0, \dots 0, v_j, 0 \dots 0)$ where the j-th coordinate is v_j. Then $1_W \otimes \phi_j : W \otimes_K V_j \to W \otimes (\bigoplus_{i=1}^n V_j)$ by $1_W \otimes \phi_j(w \otimes v_j) = (1_W w \otimes \phi_j v_j) = (w \otimes (0 \dots 0, v_j, 0 \dots 0))$ Now we construct the inverse:

$$h : \bigoplus_{i=1}^n (W \otimes_K V_i) \to W \otimes_K \left(\bigoplus_{i=1}^n V_i \right)$$
$$(w \otimes v_i) \mapsto w \otimes \sum_{i=1}^n \phi_i v_i$$

We need to check that h is inverse to f and that h defines a homomorphism (h is a homomorphism by universal property of $\oplus$). Partial check:

$$f(h((w \otimes v_i))) = f((w \otimes \sum \phi_i v_i)) = f(w \otimes (v_i)) = (w \otimes v_i)$$
$$h(f(w \otimes (v_i))) = h((w \otimes v_i)) = w \otimes \sum \phi_i v_i = w \otimes (v_i)$$

Prop.4 Let V, W be K-vector spaces with $\dim V = 1$. Then $W \otimes V \cong W$.

Pf. Let $\{v\}$ be a basis for V. Elements of $W \otimes V$ are of the form $\sum_{i=1}^n w_i \otimes v_i, w_i \in W, v_i \in V$ Since $\{v\}$ is a basis for V we can write each $v_i = a_i v$ for $a_i \in K$. Then for each term we have $w_i \otimes v_i = w_i \otimes a_i v = a_i w_i \otimes v$. Then we can define a bilinear map

$$W \times V \to W$$
$$(w, av) \mapsto aw$$

which, by the property of tensor product, induces a homomorphism

$$\begin{aligned} f: W\otimes V &\to W \\ w\otimes av &\mapsto aw. \end{aligned}$$

Note that we also have a linear map:

$$\begin{aligned} g: W &\to W\otimes V \\ w &\mapsto w\otimes v \end{aligned}$$

which is inverse to f. Check: $g(f(w\otimes av)) = g(aw) = aw\otimes v = w\otimes av$. Therefore $W\otimes V \cong W$.

Prop.5 Let V, W be K-vector spaces with bases $B_V = \{v_1, \dots v_n\}, B_W = \{w_1, \dots, w_m\}$ respectively. Then $B_{V\otimes W} = \{v_i\otimes w_j : 1\le i\le n, 1\le j\le m\}$ is a basis for $V\otimes W$ and $\dim(V\otimes W) = nm$.

Pf. We are given that $V = Kv_1\oplus\cdots\oplus Kv_n, W = Kw_1\oplus\cdots\oplus Kw_m$. So that we can compute:

$$\begin{aligned} W\otimes V =& W\otimes(Kv_1\oplus\cdots\oplus Kv_n)\cong(W\otimes Kv_1)\oplus\cdots\oplus(W\otimes Kv_1) \\ &(\text{Prop.4 and dimension formula for } \oplus\ )\Rightarrow \dim(V\otimes W) = \dim(W)+\cdots+\dim(W) = nm \end{aligned}$$

Now we show that $B_{V\otimes W}$ spans $V\otimes W$. A general element is of the form $\sum_i^l (x_i\otimes y_i), x_i\in V, y_i\in W$.

$$\begin{aligned} x_i\otimes y_i &= (a_1v_1+\cdots+a_nv_n)\otimes(b_1w_1+\cdots+b_mw_m) \\ &= a_1v_1\otimes(b_1w_1+\cdots+b_mw_m)+\cdots+a_nv_n\otimes(b_1w_1+\cdots+b_mw_m) \\ &= a_1b_1v_1\otimes w_1+\cdots+a_1b_mv_1\otimes w_m+\cdots+a_nb_1v_n\otimes w_1+\cdots+a_nb_mv_n\otimes w_m \\ &= \sum_{1\le i\le n, 1\le j\le m} a_ib_jv_i\otimes w_j \in \langle B_{V\otimes W}\rangle \end{aligned}$$

Note: $B_{V\otimes W}$ is often considered in dictionary order.