Tensor Products and the Trace Map

Linear Algebra

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Recall the definition of the tensor product.

Definition 1 (Lipschutz, Lipson)**.** *Let V and W be vector spaces over a field K. The tensor product of V and W is a vector space, $V \otimes W$, over K with a bilinear map $\psi : V \times W \to V \otimes W$ such that for any K-vector space, U, and a bilinear map $f : V \times W \to U$ there exists a unique linear map $f^* : V \otimes W \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} V \times W & \xrightarrow{f} & U \\ \psi\big\downarrow & \nearrow_{f^*} & \\ V \otimes W & & \end{array}$$

We begin with several examples.

Example 1. *Let $V = \mathbb{R}^n_m$ and $W = \mathbb{R}^q_p$. Then $V \otimes W \cong \mathbb{R}^{nq}_{mp}$ such that for $A \in V$ with $A = [a_{ij}]$ and $B \in W$ then $A \otimes B = [a_{ij}B]$. For example, let $V = W = \mathbb{R}^2_2$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then*

$$A \otimes B = \begin{bmatrix} 1\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Example 2 (Halmos)**.** *Let $P_n(x)$ be the vector space of polynomials in the indeterminate x of degree less than n with real coefficients. Let $P_m(y)$ be the vector space of polynomials in the indeterminate y of degree less than m with real coefficients. Now let $P_{n,m}(x, y)$ be the vector space of polynomials in the two indeterminates x and y with real coefficients such that the degree of any polynomial with respect to x is less than n and the degree of any polynomial with respect to y is less than m.*

We claim that $P_n(x) \otimes P_m(y) \cong P_{n,m}(x, y)$. First define a function $F : P_n(x) \times P_m(y) \to P_{n,m}(x, y)$ by $F(f(x), g(y)) = f(x)g(y)$. Note that F is bilinear. Then by the definition of the tensor product there exists a unique linear map F^* such that the following diagram commutes:

$$\begin{array}{ccc} P_n(x) \times P_m(y) & \xrightarrow{F} & P_{n,m}(x, y) \\ \psi\big\downarrow & \nearrow_{F^*} & \\ P_n(x) \otimes P_m(y) & & \end{array}$$

Furthermore, note that

$$F^*(f(x) \otimes g(y)) = F^*(\psi(f(x), g(y))) = F(f(x), g(y)) = f(x)g(y).$$

We now claim that F^* is surjective. Let $h(x, y) \in P_{n,m}(x, y)$. Then $h(x, y) = \sum_{i,j} a_{ij} x^i y^j$. For each

$0 \leq i \leq n-1$ and $0 \leq j \leq m-1$ we have

$$F^*\left(\sum_{i,j} a_{ij}x^i \otimes y^j\right) = \sum_{i,j} a_{ij}F^*(x^i \otimes y^j) = \sum_{i,j} a_{ij}x^i y^j = h(x,y)$$

Thus, $F^*(P_n(x) \otimes P_m(y)) = P_{n,m}(x,y)$. Since $\dim(P_n(x) \otimes P_m(y)) = nm = \dim(P_{n,m}(x,y))$ we conclude that F^* is an isomorphism of vector spaces.

We will now proceed to develop to trace map.

Proposition 1. *Let V be a finite dimensional vector space over a field K. Then $V \otimes V^* \cong End(V)$.*

Proof. Define $\Phi : V \otimes V^* \to \text{End}(V)$ such that $v \otimes f \mapsto \phi$ where $\phi : V \to V$ is defined by $\phi(w) = f(w)v$. We claim that Φ is an injection. Let $v \otimes f \in \ker \Phi$. Then $f(w)v = 0$ for all $w \in V$. Hence, either $v = 0$ or f is the zero map, $f \equiv 0$. If $v = 0$ then

$$v \otimes f = 0 \otimes f = 0(0 \otimes f) = 0 \otimes 0.$$

If $f \equiv 0$ then

$$v \otimes f = v \otimes 0 = 0(v \otimes 0) = 0 \otimes 0.$$

Thus, $v \otimes f = 0 \otimes 0$. We conclude that $\ker \Phi$ is trivial and thus, Φ is injective. Since $\dim(V \otimes V^*) = \dim(V)^2 = \dim(\text{End}(V))$ we conclude that Φ is an isomorphism of vector spaces. We leave it as an exercise to show that Φ is linear. □

Definition 2. *Given V, a finite dimensional K-vector space, there is a bilinear map $EV : V \times V^* \to K$ given by $EV(v,f) = f(v)$. Thus, by the definition of the tensor product, There exists a unique linear map, call it ev, such that the following diagram commutes:*

$$\begin{array}{ccc} V \times V^* & \xrightarrow{\text{EV}} & K \\ \psi \downarrow & \nearrow_{\text{ev}} & \\ V \otimes V^* & & \end{array}$$

The map ev is called the evaluation map. Therefore, the evaluation map is the following linear map:

$$\begin{aligned} ev : V \otimes V^* &\to K \\ v \otimes f &\mapsto f(v) \end{aligned}$$

Proposition 2. *Let V be a finite dimensional K-vector space, let tr be the trace map on $End(V)$ and let Φ be the isomorphism from the proof of Proposition (1). Recall that*

$$\begin{aligned} \Phi : V \otimes V^* &\to End(V) \\ v \otimes f &\mapsto [w \mapsto f(w)v] \end{aligned}$$

Then, for any $v \otimes f \in V \otimes V^*$,

$$ev(v \otimes f) = tr(\Phi(v \otimes f))$$

Proof. Let $\{e_1, \ldots, e_n\}$ a basis for V and let $\{e^1, \ldots, e^n\}$ be the corresponding dual basis of V^*. Recall that a basis along with its dual basis satisfies the Kronecker delta function, i.e. $e^j(e_i) = \delta_{i,j}$ for each $i, j = 1, \ldots, n$. As proved by Lemann, $\{e_i \otimes e^j\}_{i,j}$ is then a basis for $V \otimes V^*$.

Observe that for any $v \otimes f \in V \otimes V^*$, there exist coefficients, $\alpha_{ij} \in K$ which allow us to uniquely express

$$v \otimes f = \sum_{i,j} \alpha_{ij} e_i \otimes e^j$$

Note that $\Phi(v \otimes f) = \phi : V \to V$ where ϕ is defined on basis vectors as

$$\phi(e_l) = \sum_{i,j} \alpha_{ij} e^j(e_l) e_i = \sum_{i,j} \alpha_{ij}(\delta_{l,j}) e_i = \sum_i \alpha_{il} e_i$$

Therefore, the matrix representing $\Phi(v \otimes f) = \phi$ with respect to $\{e_1, \ldots, e_n\}$ is $[\alpha_{ij}]$. Note that $\operatorname{tr}([\alpha_{ij}]) = \sum_i \alpha_{ii}$.

Additionally,

$$\operatorname{ev}(v \otimes f) = \operatorname{ev}\left(\sum_{i,j} \alpha_{ij} e_i \otimes e^j\right) = \sum_{i,j} \alpha_{ij} \operatorname{ev}(e_i \otimes e^j) = \sum_{i,j} \alpha_{ij} \delta_{i,j} = \sum_i \alpha_{ii}$$

Thus,

$$\operatorname{ev}(v \otimes f) = \sum_i \alpha_{ii} = \operatorname{tr}([\alpha_{ij}]) = \operatorname{tr}(\Phi(v \otimes f))$$

as desired. □

Warning! in the following examples we abuse notation to conflate matrices with the endomorphisms they represent.

Example 3. *Let $V = \mathbb{R}^2$. Then endomorphisms of V are 2×2 matrices. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Observe that $Ae_1 = e_1$ and $Ae_2 = 0$. Let $v = e_1$ and let $f : V \to \mathbb{R}$ be defined by $f(e_1) = 1$ and $f(e_2) = 0$.*

We first claim that $\Phi(v \otimes f) = A$. Consider that

$$\Phi(v \otimes f)(e_1) = f(e_1)e_1 = e_1.$$

We now compute

$$\Phi(v \otimes f)(e_2) = f(e_2)e_1 = 0e_1 = 0.$$

Thus, $\Phi(v \otimes f) = A$. Lastly, consider that $tr(A) = 1$ and that $ev(v \otimes f) = f(v) = 1$ as desired.

Example 4. *Again let $V = \mathbb{R}^2$ and consider the identity endomorphism, given by $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define $v_1, v_2 \in V$ and $f_1, f_2 \in V^*$ as follows:*

$$\begin{aligned} v_1 &= e_1 & f_1(e_1) &= 1 \\ & & f_1(e_2) &= 0 \\ v_2 &= e_2 & f_2(e_1) &= 0 \\ & & f_2(e_2) &= 1 \end{aligned}$$

We first claim that

$$\Phi(v_1 \otimes f_1 + v_2 \otimes f_2) = I$$

Observe that

$$\Phi(v_1 \otimes f_1 + v_2 \otimes f_2)(e_1) = \Phi(v_1 \otimes f_1)(e_1) + \Phi(v_2 \otimes f_2)(e_1) = f_1(e_1)e_1 + f_2(e_1)e_2 = 1e_1 + 0e_2 = e_1$$

and that

$$\Phi(v_1 \otimes f_1 + v_2 \otimes f_2)(e_2) = \Phi(v_1 \otimes f_1)(e_2) + \Phi(v_2 \otimes f_2)(e_2) = f_1(e_2)e_1 + f_2(e_2)e_2 = 0e_1 + 1e_2 = e_2$$

Thus, $\Phi(v_1 \otimes f_1 + v_2 \otimes f_2) = I$. Lastly, observe that

$$tr(I) = 2$$

and that

$$ev(v_1 \otimes f_1 + v_2 \otimes f_2) = ev(v_1 \otimes f_1) + ev(v_2 \otimes f_2) = f_1(e_1) + f_2(e_2) = 1 + 1 = 2.$$

Example 5. *Let* $V = \mathbb{R}^2$ *and* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Define* $v_1, v_2 \in V$ *and* $f_1, f_2 \in V^*$ *as follows:*

$$v_1 = \begin{bmatrix} a \\ c \end{bmatrix} \qquad f_1(e_1) = 1$$

$$f_1(e_2) = 0$$

$$v_2 = \begin{bmatrix} b \\ d \end{bmatrix} \qquad f_2(e_1) = 0$$

$$f_2(e_2) = 1$$

We first claim that

$$\Phi(v_1 \otimes f_1 + v_2 \otimes f_2) = A$$

Observe that

$$\Phi(v_1 \otimes f_1 + v_2 \otimes f_2)(e_1) = \Phi(v_1 \otimes f_1)(e_1) + \Phi(v_2 \otimes f_2)(e_1) = f_1(e_1)v_1 + f_2(e_1)v_2 = v_1 = Ae_1$$

and that

$$\Phi(v_1 \otimes f_1 + v_2 \otimes f_2)(e_2) = \Phi(v_1 \otimes f_1)(e_2) + \Phi(v_2 \otimes f_2)(e_2) = f_1(e_2)v_1 + f_2(e_2)v_2 = v_2 = Ae_2$$

Thus, $\Phi(v_1 \otimes f_1 + v_2 \otimes f_2) = A$. *Lastly, observe that*

$$tr(A) = a + d$$

and that

$$\begin{aligned} ev(v_1 \otimes f_1 + v_2 \otimes f_2) &= ev(v_1 \otimes f_1) + ev(v_2 \otimes f_2) \\ &= f_1(v_1) + f_2(v_2) \\ &= f_1(ae_1 + ce_2) + f_2(be_1 + de_2) \\ &= af_1(e_1) + cf_1(e_2) + bf_2(e_1) + df_2(e_2) \\ &= a + d. \end{aligned}$$