

- (1) (12 Points) Let $V = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$, so V is a vector space over the field of reals, $\mathbb{R}$. Prove that $W = \{f \in V \mid f(2) = 0\}$ is a subspace of V .

- (2) (18 Points) Answer (with brief justification) each question separately.

- (a) If $S \subseteq T \subseteq V$ and S is dependent, what is the most you can say about T ?
- (b) If $S \subseteq T \subseteq V$ and T is independent, what is the most you can say about S ?
- (c) If $T = \{w_1, w_2, \dots, w_m\} \subseteq W$ and $w \in \langle T \rangle$, what is the most you can say about $T \cup \{w\} = \{w_1, \dots, w_m, w\}$?
- (d) If $S = \{v_1, \dots, v_n\} \subseteq V$ is independent and $v \in V$ but $v \notin \langle S \rangle$, what is the most you can say about $S \cup \{v\} = \{v_1, \dots, v_n, v\}$?
- (e) If $A \in \mathbb{F}_n^n$ is row equivalent to the identity matrix, what is the most you can say about the associated linear map $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ defined by $L_A(X) = AX$?
- (f) Let $A \in \mathbb{F}_n^m$ with $m < n$. What is the most you can say about the number of free variables in the solutions to the linear system $AX = 0_1^m$?

- (3) (20 Points) Let $W = \langle S \rangle$ be the span of S , where S is the following subset of $\mathbb{R}_2^2$,
- $$S = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}, \begin{bmatrix} 3 & -5 \\ -6 & 8 \end{bmatrix}, \begin{bmatrix} -3 & 11 \\ 7 & -13 \end{bmatrix} \right\}.$$

- (a) Show that S is dependent and find the nontrivial dependence relations which allow redundant vectors to be removed.
- (b) Find conditions on a, b, c, d such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$. Is $W = \mathbb{R}_2^2$?
- (c) Find a basis for W .

- (4) (10 points) Answer each question separately. Let A^T mean the transpose of A .

- (a) Suppose $0_9^4 \neq A \in \mathbb{F}_9^4$ and $L_A : \mathbb{F}^9 \rightarrow \mathbb{F}^4$ is the linear map $L_A(X) = AX$. What are the possibilities for $\dim(\text{Ker}(L_A))$?
- (b) If $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, find $\det(A^3 B^{-1} C^T)$.
- (c) If $S = \{v_1, \dots, v_n\}$ is a basis of V and $L : V \rightarrow W$, what is the most you can say about the set $L(S) = \{L(v_1), \dots, L(v_n)\}$?
- (d) Let $A \in \mathbb{F}_n^m$ so $L_A(X) = AX$ defines $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$. If C is the RREF of A , which columns of A form a basis for $\text{Range}(L_A)$?
- (e) If $A \in \mathbb{F}_n^n$ and $\alpha \in \mathbb{F}$, find $\det(\alpha A)$.

- (5) (15 Points) Let $L : \mathbb{R}_2^2 \rightarrow \mathbb{R}^2$ be the linear map given by $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2b - c - 2d \\ -a + b + c - d \end{bmatrix}$ and let S and T be the standard bases of $\mathbb{R}_2^2$ and $\mathbb{R}^2$, respectively. Let other ordered bases be $S' = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

- (a) (2 points) Find the matrix ${}_T[L]_S$ representing L from S to T .
 (b) (3 points) Find a basis for $\text{Ker}(L)$.
 (c) (3 points) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' directly (without using transition matrices).
 (d) (4 points) Find the transition matrices ${}_SP_{S'}$ and ${}_{T'}Q_T$.
 (e) (3 point) Compute the product ${}_{T'}Q_T {}_T[L]_S {}_SP_{S'}$. Compare it to part (c).

6. (25 Points) Let $A = \begin{bmatrix} 4 & 0 & 3 & 0 \\ 0 & 4 & 0 & 3 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$.

- (a) (7 Pts)
 (1) Find the **characteristic polynomial** of A .
 (2) Find the **eigenvalues** of A .
 (3) Find their **algebraic multiplicities**.
 (b) (18 points) The **characteristic polynomial** of B is $(t - 2)^3 (t + 2)$ and B can be diagonalized.
 (1) **Find a basis** $\mathcal{B}$ of $\mathbb{R}^4$ consisting of eigenvectors for B .
 (**Hint:** $[1 \ 1 \ 1 \ 1]^T$ is an eigenvector of B with eigenvalue -2 .)
 (2) Find a **diagonal matrix** D similar to B .
 (3) Find an **invertible** matrix P such that $D = P^{-1}BP$.

1. (12 Points) Let $V = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$, so V is a vector space over the field of reals, $\mathbb{R}$. Prove that $W = \{f \in V \mid f(2) = 0\}$ is a subspace of V .

SOLUTION: To show W is a subspace, show three facts: the zero function is in W and that W is closed under addition and scalar multiplication. The zero function defined by $\theta(t) = 0$ for all $t \in \mathbb{R}$, is in W because $\theta(t) = 0$ so $\theta(2) = 0$. If $f, g \in W$, then $f + g \in W$ because $(f + g)(t) = f(t) + g(t)$ so $(f + g)(2) = f(2) + g(2) = 0 + 0 = 0$. If $f \in W$ and $b \in \mathbb{R}$ then $bf \in W$ since $(bf)(2) = b(f(2)) = b(0) = 0$.

2. (18 points)
- (a) T is also a dependent set. (A set containing a dependent set is dependent.)
 - (b) S is independent. (A subset of an independent set is independent.)
 - (c) $T \cup \{w\}$ is dependent. (Last vector is a linear combo of previous vectors.)
 - (d) $S \cup \{v\}$ is independent. (No vector is a linear combo of previous vectors.)
 - (e) A is invertible, so L_A is invertible, bijective, and has inverse $L_{A^{-1}}$.
 - (f) Since $m < n$ the linear system has more variables than equations, so it must have nontrivial solutions. When the augmented matrix is row reduced, it can have at most m leading ones, so there are at least $n - m$ columns without leading ones, giving at least $n - m$ free variables in the solution space.

3. (20 points) (a) Let the 5 vectors of S be denoted $v_1, \dots, v_5$. Determine if $\sum_{i=1}^5 a_i v_i = \theta$ has nontrivial solutions. Reduce

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 3 & -3 & 0 \\ 1 & -1 & 3 & -5 & 11 & 0 \\ -1 & -2 & 2 & -6 & 7 & 0 \\ 1 & 2 & -4 & 8 & -13 & 0 \end{array} \right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{array}{l} x_1 = -s \\ x_2 = -2r + s \\ x_3 = r - 3s \\ x_4 = r \in \mathbb{R} \\ x_5 = s \in \mathbb{R} \end{array}$$

which has nontrivial solutions so S is dependent. Each free variable gives a dependence relation. When $r = 1$ and $s = 0$, we get $-2v_2 + v_3 + v_4 = \theta$ and when $r = 0$ and $s = 1$ we get $-v_1 + v_2 - 3v_3 + v_5 = \theta$. The redundant vectors v_4 and v_5 are linear combinations of v_1, v_2 and v_3 , so $S = \langle v_1, v_2, v_3 \rangle$.

- (b) Since $W = \langle v_1, v_2, v_3 \rangle$, a matrix is in this span when the following system is consistent:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 1 & -1 & 3 & b \\ -1 & -2 & 2 & c \\ 1 & 2 & -4 & d \end{array} \right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a + c \\ 0 & 1 & 0 & -a - \frac{3}{2}c - \frac{1}{2}d \\ 0 & 0 & 1 & \frac{1}{2}c - \frac{1}{2}d \\ 0 & 0 & 0 & -3a + b - c + d \end{array} \right]$$

which is consistent iff $0 = -3a + b - c + d$. This is the condition required, and it is not true for all matrices so $W \neq \mathbb{R}_2^2$.

- (c) From part (a), after removing the redundant vectors from S , we have $\{v_1, v_2, v_3\}$ is a basis for W . Using $0 = -3a + b - c + d$ from part (b), any element of W is

$$\begin{bmatrix} a & b \\ c & 3a - b + c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \text{ so } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is a basis of W .

4. (10 points)

- (a) Since $0_9^4 \neq A \in \mathbb{F}_9^4$, A row reduces to a RREF with r leading 1's, and $1 \leq r \leq 4$ so $9 - r \in \{5, 6, 7, 8\}$ is number of possible free variables.
- (b) $\det(A^3 B^{-1} C^T) = \frac{\det(A)^3 \det(C)}{\det(B)} = \frac{(2^3)(5)}{3} = \frac{40}{3}$.
- (c) The set $L(S) = \{L(v_1), \dots, L(v_n)\}$ spans $\text{Range}(L)$. It will only be a basis for $\text{Range}(L)$ when L is injective.
- (d) The pivot columns of A , corresponding to columns of C having a leading 1, form a basis for $\text{Range}(L_A)$.
- (e) If $A \in \mathbb{F}_n^n$ and $\alpha \in \mathbb{F}$, then $\det(\alpha A) = \alpha^n \det(A)$.

(5) (15 Points) Let $L : \mathbb{R}_2^2 \rightarrow \mathbb{R}^2$ be the linear map given by $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2b - c - 2d \\ -a + b + c - d \end{bmatrix}$ and let S and T be the standard bases of $\mathbb{R}_2^2$ and $\mathbb{R}^2$, respectively. Let other ordered bases be $S' = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(a) (2 points) The matrix ${}_T[L]_S = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ by row reducing $[T|L(S)]$.

(b) (3 points) To find $\text{Ker}(L)$ row reduce $\begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ -1 & 1 & 1 & -1 & 0 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$ giving solutions $a = c$, $b = d$ with c and d free variables. Thus,
 $\text{Ker}(L) = \left\{ \begin{bmatrix} c & d \\ c & d \end{bmatrix} \in \mathbb{R}_2^2 \mid c, d \in \mathbb{R} \right\}$, which has basis $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

(c) (3 points) ${}_{T'}[L]_{S'} = \begin{bmatrix} 0 & -3 & -6 & -3 \\ 0 & 2 & 3 & 1 \end{bmatrix}$ since we row reduce

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 & -3 & -2 \\ 1 & 2 & 0 & 1 & 0 & -1 \end{array} \right]_{\substack{T' \\ L(S')}} \quad \text{to} \quad \left[\begin{array}{cc|cc} 1 & 0 & 0 & -3 & -6 & -3 \\ 0 & 1 & 0 & 2 & 3 & 1 \end{array} \right]_{\substack{I_2 \\ {}_{T'}[L]_{S'}}}$$

(d) (4 points) The transition matrices ${}_S P_{S'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and ${}_T Q_{T'} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ since

$$S \text{ and } T \text{ are the standard bases. Then } {}_{T'} Q_T = ({}_T Q_{T'})^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

(e) (3 point) The matrix product

$$\begin{aligned} {}_{T'} Q_T {}_T [L]_S {}_S P_{S'} &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & -2 \\ -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & -3 & -3 \\ -2 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -6 & -3 \\ 0 & 2 & 3 & 1 \end{bmatrix} \end{aligned}$$

equals the answer to part (c) as it should.

6. (25 points)

(a) (7 points)

Solutions: (1) Doing elementary row operations, we have $\det(A - tI_4) =$

$$\det \begin{bmatrix} 4-t & 0 & 3 & 0 \\ 0 & 4-t & 0 & 3 \\ 3 & 0 & 4-t & 0 \\ 0 & 3 & 0 & 4-t \end{bmatrix} = \det \begin{bmatrix} 1-t & 0 & t-1 & 0 \\ 0 & 1-t & 0 & t-1 \\ 3 & 0 & 4-t & 0 \\ 0 & 3 & 0 & 4-t \end{bmatrix} =$$

$$(t-1)^2 \det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & 4-t & 0 \\ 0 & 3 & 0 & 4-t \end{bmatrix} = (t-1)^2 \det \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 7-t & 0 \\ 0 & 0 & 0 & 7-t \end{bmatrix}$$

$$= (t-1)^2 (t-7)^2$$

(2) The **eigenvalues** of A are $\lambda_1 = 1$ and $\lambda_2 = 7$

(3) The corresponding **algebraic multiplicities** are $k_1 = 2$ and $k_2 = 2$.

(b) (18 points)

Solutions: The eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = -2$ with algebraic multiplicity $k_2 = 1$. (1) Check the $\lambda_1 = 2$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = 2$ to $B - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{array}{l} x_1 = -r - s - t \\ x_2 = r \in \mathbb{R} \\ x_3 = s \in \mathbb{R} \\ x_4 = t \in \mathbb{R} \end{array}, \text{ then}$$

$$B_{\lambda_1} = \left\{ \begin{bmatrix} -r - s - t \\ r \\ s \\ t \end{bmatrix} \in \mathbb{R}^4 \mid r, s, t \in \mathbb{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Now find the $\lambda_2 = -2$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = -2$ to $B - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & -1 & -1 & 3 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{array}{l} x_1 = r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbb{R} \end{array}, \text{ then}$$

$$B_{\lambda_2} = \left\{ \begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} \in \mathbb{R}^4 \mid r \in \mathbb{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ as given in the hint.}$$

An eigen-basis of $\mathbb{R}^4$ for B is $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$(2) \ D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad (3) \ P = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
