Math 507 Linear Algebra and Matrix Theory Fall 2025 Final Exam Feingold

Show work to justify your answers. A^T **denotes the transpose of matrix** A**.**

(1) (15 Points) Let $L : \mathbb{F}_2^2 \to \mathbb{F}_2^2$ be the linear transformation given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} (a-b-c) & (a-c+d) \\ (b+c-d) & (2a-c) \end{bmatrix}$$

(a) (2 points) Find the set of **all vectors** in Ker(L).
(b) (2 points) Find a **basis** for Ker(L) and find dim(Ker(L)).
(c) (1 points) Is L injective? **Explain why!**
(d) (3 points) Find **all vectors** in Range(L).
(e) (3 points) Find a **basis** for Range(L) and find dim(Range(L)).
(f) (2 points) Is L onto? **Explain why!**
(g) (2 points) Is L invertible? **Explain why!** If so, find a formula for L^{-1}.

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(2) (20 Points, 4 pts each) Answer each question separately **with a brief justification**.

(a) Let $L : V \to W$ and let $S = \{v_1, \cdots, v_n\}$ be a basis for V. If L is **injective**, what is the **most** you can say about the set of vectors $L(S) = \{L(v_1), \cdots, L(v_n)\}$?

(b) If $L : \mathbb{F}_4^3 \to \mathbb{F}^7$, find the best bounds $a \leq \dim(Ker(L)) \leq b$.

(c) If $L : \mathbb{F}^3 \to \mathbb{F}^7$, find the best bounds $c \leq \dim(Range(L)) \leq d$.

(d) Let $L : V \to V$ where $V = W_1 \oplus W_2$ for L-invariant subspaces, W_1 and W_2, and let L_1 and L_2 be the restrictions of L to these two subspaces. What is the relationship between the $char_L(t)$, $char_{L_1}(t)$ and $char_{L_2}(t)$?

(e) Continuing (d), what is the relationship between the minimal polynomials $m_L(t)$, $m_{L_1}(t)$ and $m_{L_2}(t)$?

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(3) (20 Points, 4 pts each) Answer each question separately with a brief justification. V is a vector space over a field $\mathbb{F}$.

(a) If $S = \{v_1, \ldots, v_n\} \subseteq V$ is **independent** and $v_{n+1} \in \langle S \rangle$, and $T = \{v_1, \ldots, v_n, v_{n+1}\}$, what is the **most** you can say about $dim(\langle T \rangle)$?

(b) If $S = \{v_1, \ldots, v_m\} \subseteq V$ and $v_{m+1} \in V$ but $v_{m+1} \notin \langle S \rangle$ then what is the **most** you can say about $dim(\langle S \cup \{v_{m+1}\} \rangle)$?

(c) Let $A \in \mathbb{F}_n^m$ with $rank(A) = r$. What is the **most** you can say about $dim(Ker(L_A))$?

(d) If $A \in \mathbb{F}_n^n$ has $rank(A) = n$, what is the **most** you can say about $L_A : \mathbb{F}^n \to \mathbb{F}^n$?

(e) If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, what is the **most** you can say about the set $\{[v_1]_S, [v_2]_S, \ldots, [v_n]_S\}$ of coordinates of vectors in S with respect to S?

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(4) (15 Pts) Let $V = \mathbb{R}^4$ with the standard dot product, and let $W = S^\perp$, the orthogonal complement in V of $S = \{u_1 = \begin{bmatrix} 1 & 2 & 1 & 3 \end{bmatrix}^T, u_2 = \begin{bmatrix} 2 & 3 & 1 & 1 \end{bmatrix}^T\}$.
Let $T = \{w_1, w_2\}$ be the basis for W obtained by solving $u_i \cdot X = 0$ for $0 \leq i \leq 2$.

(a) (10 pts) Use Gram-Schmidt to get an **orthogonal** basis $T' = \{w_1', w_2'\}$ for W.

(b) (5 pts) Use T' to find the coefficients x_i of the projection $Proj_W(v) = x_1 w_1' + x_2 w_2'$ of the general vector $v = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \in V$ into W.

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(5) (15 Points) For the real matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ do the following.

(a) Find the **characteristic polynomial** $char_A(t)$, the **eigenvalues** of A, and their **algebraic multiplicities**.

(b) Find the **minimal polynomial** $m_A(t)$. What does it tell you?

(c) Find the eigenspaces of A, their bases and geometric multiplicities.

(d) If A can be diagonalized, find diagonal matrix D and transition matrix P such that $D = P^{-1}AP$. Otherwise, find the **Jordan Canonical Form** matrix J similar to A.

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(6) (15 Points) Suppose that $A \in \mathbb{R}^{12}_{12}$ has **characteristic** and **minimal** polynomials

$$char_A(t) = (t-5)^7(t-8)^5 \quad \text{and} \quad m_A(t) = (t-5)^4(t-8)^3.$$

Find all possible **Jordan canonical form** matrices J to which A might be similar, but not to each other, and for each one give the pair of geometric multiplicities (g_1, g_2).

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(7) (15 Points) Suppose that $A \in \mathbb{R}^{10}_{10}$ has **characteristic** and **minimal** polynomials

$$char_A(t) = (t^2+2t+3)^5 \quad \text{and} \quad m_A(t) = (t^2+2t+3)^2.$$

Find all possible **Rational Canonical Form** matrices R to which A might be similar, but not to each other. In each case, how many **cyclic subspaces** occur in the corresponding decomposition of $\mathbb{R}^{10}$?

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(8) (15 Pts) Let $L : \mathbb{R}^2_2 \to \mathbb{R}^2$ be the linear map $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b+d \\ 3a+2b-c \end{bmatrix}$ and let S and T be the standard bases of $\mathbb{R}^2_2$ and $\mathbb{R}^2$, respectively. Let other ordered bases be $S' = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

(a) (2 pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

(b) (3 pts) Find a basis for $\text{Ker}(L)$.

(c) (4 pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' directly (without using transition matrices) by row reducing $[T'|L(S')]$.

(d) (4 pts) Find the transition matrices ${}_SP_{S'}$ and ${}_{T'}Q_T$.

(e) (2 pts) Compute the product ${}_{T'}Q_T \; {}_T[L]_S \; {}_SP_{S'}$. Compare it to part (c).

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(9) (10 Pts) Fix $M \in \mathbb{F}^n_n$ and let $U = \{A \in \mathbb{F}^n_n \mid A^TM = -MA\}$ where A^T is A transpose.

(a) (5 pts) Prove that U is a subspace of $\mathbb{F}^n_n$.

(b) (5 pts) Prove that for any $A, B \in U$ we have $AB - BA \in U$.

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(10) (10 Pts) Let V be the real inner product space with basis $S = \{v_1, v_2\}$ and inner product $(v, w) = [v]^T_S M [w]_S$ where $M = [(v_i, v_j)] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Let $L : V \to V$ be a linear map represented by a matrix $A = {}_S[L]_S$ so $A[v]_S = [L(v)]_S$ for all $v \in V$.

(a) (5 Pts) What condition on A means that L an orthogonal map with respect to this inner product, that is, $(L(v), L(w)) = (v, w)$ for all $v, w \in V$? Justify your answer.

(b) (5 Pts) Using your answer to part (a), determine whether the map $L(a_1v_1 + a_2v_2) = -a_2v_1 + a_1v_2$ is orthogonal with respect to $(\cdot,\cdot)$ on V.

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(1) (15 points) (a) (2 points) To find all vectors in Ker(L), row reduce

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 2 & 0 & -1 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} a = r \\ b = -r \\ c = 2r \\ d = r \in \mathbb{F} \end{array} \text{ so } \begin{array}{c} Ker(L) = \\ \left\{ \begin{bmatrix} r & -r \\ 2r & r \end{bmatrix} \;\middle|\; r \in \mathbb{F} \right\}. \end{array}$$

(b) (2 points) Ker(L) has basis $\left\{ \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \right\}$ and $\dim(Ker(L)) = 1$.

(c) (1 points) L is not injective since Ker(L) is non-trivial.

(d) (3 points) All vectors: $Range(L) = \left\{ \begin{bmatrix} (a-b-c) & (a-c+d) \\ (b+c-d) & (2a-c) \end{bmatrix} \in \mathbb{F}^2_2 \;\middle|\; a, b, c, d \in \mathbb{F} \right\}$.

Consistency condition: $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in Range(L)$ iff $0 = w + x + y - z$.

(e) (3 points) $Range(L)$ is spanned by the set of four vectors

$$\left\{ v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

and the Ker(L) being non-trivial means they are dependent. The basis vector of Ker(L) gives the dependence relation $v_1 - v_2 + 2v_3 + v_4 = \theta$, so $v_4 = -v_1 + v_2 - 2v_3$ is redundant. A basis for Range(L) is then $\{v_1, v_2, v_3\}$ and dim(Range(L)) = 3.

(f) (2 points) L is not onto since $\dim(Range(L)) = 3 < 4 = \dim(\mathbb{F}^2_2)$.

(g) (2 points) L is not invertible since it is not bijective.

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(2) (20 Points, 4 pts each) Answer (with brief justification) each question separately.

(a) For $L : V \to W$ injective and $S = \{v_1, \cdots, v_n\}$ a basis for V, we can say that $L(S) = \{L(v_1), \cdots, L(v_n)\}$ is a **basis** of $Range(L)$.

(b) Since $L : \mathbb{F}^3_4 \to \mathbb{F}^7$, we know $12 = \dim(\mathbb{F}^3_4) = \dim(Ker(L)) + \dim(Range(L))$. But $0 \le \dim(Range(L)) \le 7$, so $5 \le \dim(Ker(L)) \le 12$.

(c) Since $L : \mathbb{F}^3 \to \mathbb{F}^7$, we know $3 = \dim(\mathbb{F}^3) = \dim(Ker(L)) + \dim(Range(L))$. But $0 \le \dim(Ker(L)) \le 3$, so $0 \le \dim(Range(L)) \le 3$.

(d) $L : V \to V$ where $V = W_1 \oplus W_2$ for L-invariant subspaces, W_1 and W_2, and $L_i = L|_{W_i}$. If $B = B_1 \cup B_2$ is a basis of V where B_i is a basis of W_i, then ${}_B[L]_B = diag({}_{B_1}[L_1]_{B_1}, {}_{B_2}[L_2]_{B_2})$ is block diagonal, so $char_L(t) = char_{L_1}(t) char_{L_2}(t)$ since the characteristic polynomial of a block diagonal matrix is the product of the characteristic polynomials of each block.

(e) Continuing (d), the relationship of the minimal polynomials $m_L(t) = lcm(m_{L_1}(t), m_{L_2}(t))$.

(3) (20 Points, 4 pts each) Answer (with brief justification) each question separately.
SOLUTIONS:

(a) If $S = \{v_1, \ldots, v_n\} \subseteq V$ is **independent** and $v_{n+1} \in \langle S \rangle$, and $T = \{v_1, \ldots, v_n, v_{n+1}\}$, then $\langle S \rangle = \langle T \rangle$ so $dim(\langle T \rangle) = dim(\langle S \rangle) = n$.

(b) If $S = \{v_1, \ldots, v_m\} \subseteq V$ and $v_{m+1} \notin \langle S \rangle$ then v_{m+1} is not a redundant vector in $S \cup \{v_{m+1}\}$ so $dim(\langle S \cup \{v_{m+1}\} \rangle) = dim(\langle S \rangle) + 1 \leq m + 1$.

(c) Let $A \in \mathbb{F}_n^m$ with $rank(A) = r$. You can say $dim(Ker(L_A)) = n - r$ since there will be $n - r$ columns without leading ones in the RREF matrix row equivalent to A.

(d) If $A \in \mathbb{F}_n^n$ has $rank(A) = n$, then $L_A : \mathbb{F}^n \to \mathbb{F}^n$ has trivial kernel, so it is **injective**, so it is **surjective**, **bijective**, **invertible** and an **isomorphism**.

(e) The set of coordinates $\{[v_1]_S, [v_2]_S, \ldots, [v_n]_S\}$ is the **standard basis** of $\mathbb{F}^n$.

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(4) (15 Points) Let $V = \mathbb{R}^4$ with the standard dot product, and let $W = S^\perp$, the orthogonal complement in V of $S = \{u_1 = [1 \quad 2 \quad 1 \quad 3]^T, u_2 = [2 \quad 3 \quad 1 \quad 1]^T\}$.
Let $T = \{w_1, w_2\}$ be the basis for W obtained by solving $u_i \cdot X = 0$ for $0 \leq i \leq 2$.

(a) (10 pts) Use Gram-Schmidt to get an **orthogonal** basis $T' = \{w_1', w_2'\}$ for W.
Solution: $W = \{X \in \mathbb{R}^4 \mid X \cdot u_i = 0, \ i = 1, 2\}$ is found by row reducing

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 3 & 1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -1 & -7 & 0 \\ 0 & 1 & 1 & 5 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r + 7s \\ x_2 = -r - 5s \\ x_3 = r \\ x_4 = s \in \mathbb{R} \end{array} \text{ so } T = \left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} 7 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{w_2} \right\}$$

Gram-Schmidt gives $w_1' = w_1$, and

$$w_2' = w_2 - \frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} w_1' = w_2 - \frac{12}{3} w_1' = \begin{bmatrix} 7 \\ -5 \\ 0 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -4 \\ 1 \end{bmatrix} \text{ so } T' = \left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{w_1'}, \underbrace{\begin{bmatrix} 3 \\ -1 \\ -4 \\ 1 \end{bmatrix}}_{w_2'} \right\}.$$

(b) (5 pts) Use T' to find the coefficients x_i of the projection $Proj_W(v) = x_1 w_1' + x_2 w_2'$ of the general vector $v = [a \quad b \quad c \quad d] \in V$ into W.
Solution: Since T' is an orthogonal basis of W, we have

$$x_i = \frac{v \cdot w_i'}{w_i' \cdot w_i'} \quad \text{so} \quad x_1 = \frac{a - b + c}{3}, \quad x_2 = \frac{3a - b - 4c + d}{27}.$$

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(5) (15 Points) (a) (3 pts) The characteristic polynomial $char_A(t)$ is $\det(tI_4 - A) = (t^2-1)^2 = (t-1)^2(t+1)^2$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$ with algebraic multiplicities $k_1 = 2$ and $k_2 = 2$.

(b) (2 pts) The minimal polynomial is $m_A(t) = (t-1)(t+1) = t^2 - 1$ since $A^2 = I_4$. A is diagonalizable because $m_A(t) = (t-1)(t+1)$ is a product of distinct linear factors.

(c) Find the eigenspaces of A, their bases and geometric multiplicities.
(3 pts) We find the λ_1-eigenspace by row reducing $[A - I_4|0_1^4] =$

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = s \\ x_2 = r \\ x_3 = r \\ x_4 = s \in \mathbb{R} \end{array}, \text{ so}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} s \\ r \\ r \\ s \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, r, s \in \mathbb{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ so } g_1 = 2.$$

(3 pts) We find the λ_2-eigenspace by row reducing $[A + I_4|0_1^4] =$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -s \\ x_2 = -r \\ x_3 = r \\ x_4 = s \in \mathbb{R} \end{array}, \text{ so}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} -s \\ -r \\ r \\ s \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, r, s \in \mathbb{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ so } g_2 = 2.$$

(d) (4 pts) Eigenbasis $T = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ gives $P = {}_SP_T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ transition matrix such that $P^{-1}AP = D = diag(1, 1, -1, -1)$.

(6) (15 Points) $char_A(t) = (t-5)^7(t-8)^5$ and $m_A(t) = (t-5)^4(t-8)^3$ so the two eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 8$ with $k_1 = 7$ and $k_2 = 5$. The powers in the minimal polynomial $m_1 = 4$ and $m_2 = 3$ tell the sizes of the largest Jordan blocks for each eigenvalue. Let

$$B = J(5,4) = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad C = J(5,3) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}, \quad D = J(5,2) = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

$$E = J(8,3) = \begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}, \quad F = J(8,2) = \begin{bmatrix} 8 & 1 \\ 0 & 8 \end{bmatrix}$$

then there are six possible Jordan canonical form matrices similar to A, corresponding to the three partitions of 7 with largest part 4, $4+3 = 4+2+1 = 4+1+1+1$, and the two partitions of 5 with largest part 3, $3+2 = 3+1+1$:

$$Diag(B, C, E, F), \quad Diag(B, D, 5, E, F), \quad Diag(B, 5, 5, 5, E, F),$$

$$Diag(B, C, E, 8, 8), \quad Diag(B, D, 5, E, 8, 8), \quad Diag(B, 5, 5, 5, E, 8, 8))$$

The corresponding pairs of geometric multiplicities (g_1, g_2) are the numbers of Jordan blocks, $(2,2)$, $(3,2)$, $(4,2)$, $(2,3)$, $(3,3)$, $(4,3)$ respectively.

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(7) (15 Points) The **characteristic and minimal polynomials** are $char_A(t) = (t^2+2t+3)^5$ and $m_A(t) = (t^2+2t+3)^2 = t^4+4t^3+10t^2+12t+9$. Define the companion matrices (4 pts, 3 pts)

$$C_1 = C((t^2+2t+3)^2) = \begin{bmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -4 \end{bmatrix}, \quad C_2 = C(t^2+2t+3) = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix}.$$

(8 pts) Then there are two possible rational canonical form matrix similar to A, $Diag(C_1, C_1, C_2)$ and $Diag(C_1, C_2, C_2, C_2)$. These are the only ways to get the given $char_A(t)$ as the product of the characteristic polynomials of each companion block, and the given minimal polynomial $m_A(t)$ as the least common multiple of the minimal polynomials of those companion blocks. In each case the number of **cyclic subspaces** occurring in the corresponding decomposition of $\mathbb{R}^{10}$ is the number of companion matrices, 3 in the first case, 4 in the second case.

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(8) (15 Pts) Let $L : \mathbb{R}^2_2 \to \mathbb{R}^2$ be the linear map $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b+d \\ 3a+2b-c \end{bmatrix}$ and let S and T be the standard bases of $\mathbb{R}^2_2$ and $\mathbb{R}^2$, respectively. Let other ordered bases be $S' = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

(a) (2 pts) Find the matrix ${}_T[L]_S$ representing L from S to T.

Solution: The matrix ${}_T[L]_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 3 & 2 & -1 & 0 \end{bmatrix}$ by row reducing $[T|L(S)]$.

(b) (3 pts) Find a basis for Ker(L).

Solution: To find $Ker(L)$ row reduce $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & -1 & 0 & 0 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix}$ giving solutions $a = c + 2d$, $b = -c - 3d$ with c and d free variables. Thus, $Ker(L) = \left\{ \begin{bmatrix} c+2d & -c-3d \\ c & d \end{bmatrix} \in \mathbb{R}^2_2 \mid c, d \in \mathbb{R} \right\}$, which has basis $\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \right\}$.

(c) (4 pts) Find the matrix ${}_{T'}[L]_{S'}$ representing L from S' to T' directly (without using transition matrices) by row reducing $[T'|L(S')]$.

Solution: We row reduce

$$\left[\begin{array}{cc|cccc} 3 & 2 & 3 & 2 & 2 & 1 \\ 1 & 1 & 4 & 4 & 5 & 3 \\ \multicolumn{2}{c|}{T'} & \multicolumn{4}{c}{L(S')} \end{array}\right] \text{ to } \left[\begin{array}{cc|cccc} 1 & 0 & -5 & -6 & -8 & -5 \\ 0 & 1 & 9 & 10 & 13 & 8 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{{}_{T'}[L]_{S'}} \end{array}\right] \text{ so } {}_{T'}[L]_{S'} = \begin{bmatrix} -5 & -6 & -8 & -5 \\ 9 & 10 & 13 & 8 \end{bmatrix}$$

(d) (4 pts) Find the transition matrices ${}_SP_{S'}$ and ${}_{T'}Q_T$.

Solution: The transition matrices ${}_SP_{S'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ since S and T are the standard bases. Then ${}_{T'}Q_T = ({}_TQ_{T'})^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

(e) (2 pts) Compute the product ${}_{T'}Q_T\ {}_T[L]_S\ {}_SP_{S'}$. Compare it to part (c).

Solution: The matrix product

$${}_{T'}Q_T\ {}_T[L]_S\ {}_SP_{S'} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 3 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -3 & 2 & 1 \\ 8 & 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -6 & -8 & -5 \\ 9 & 10 & 13 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 1 \\ 4 & 4 & 5 & 3 \end{bmatrix}$$

equals the answer to part (c) as it should.

(9) (10 Pts) Fix $M \in \mathbb{F}^n_n$ and let $U = \{A \in \mathbb{F}^n_n \mid A^T M = -MA\}$ where A^T is A transpose.

(a) (5 pts) Prove that U is a subspace of $\mathbb{F}^n_n$.

Solution: $0^n_n \in U$ since $(0^n_n)^T M = 0^n_n = -M0^n_n$. If $A, B \in U$ we have $A^T M = -MA$ and $B^T M = -MB$ so for any $a, b \in \mathbb{F}$ we have

$$(aA + bB)^T M = aA^T M + bB^T M = -aMA - bMB = -M(aA + bB)$$

so $aA + bB \in U$. Therefore, U is a subspace of $\mathbb{F}^n_n$.

(b) (5 pts) Prove that for any $A, B \in U$ we have $AB - BA \in U$.

Solution: Suppose $A, B \in U$, so $A^T M = -MA$ and $B^T M = -MB$. Then we have

$$(AB - BA)^T M = (B^T A^T - A^T B^T)M = B^T A^T M - A^T B^T M = B^T(-MA) - A^T(-MB)$$
$$= -(B^T MA - A^T MB) = -(-MBA + MAB) = M(BA - AB) = -M(AB - BA)$$

so $AB - BA \in U$.

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(10) (10 Pts) Let V be the real inner product space with basis $S = \{v_1, v_2\}$ and inner product $(v, w) = [v]^T_S M[w]_S$ where $M = [(v_i, v_j)] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Let $L : V \to V$ be a linear map represented by a matrix $A = {}_S[L]_S$ so $A[v]_S = [L(v)]_S$ for all $v \in V$.

(a) (5 Pts) What condition on A means that L an orthogonal map with respect to this inner product, that is, $(L(v), L(w)) = (v, w)$ for all $v, w \in V$? Justify your answer.

Solution: For all $v, w \in V$ the following would have to be true:

$$(L(v), L(w)) = [L(v)]^T_S M[L(w)]_S = (A[v]_S)^T MA[w]_S = [v]^T_S (A^T MA)[w]_S = [v]^T_S M[w]_S.$$

Using $v = v_i$ and $w = v_j$ in S, this says $\mathbf{e}_i^T (A^T MA)\mathbf{e}_j = \mathbf{e}_i^T M\mathbf{e}_j$ which means the (i, j) entries of matrices $A^T MA$ and M are equal, so $A^T MA = M$ is the condition on A.

(b) (5 Pts) Using your answer to part (a), determine whether the map $L(a_1 v_1 + a_2 v_2) = -a_2 v_1 + a_1 v_2$ is orthogonal with respect to $(\cdot, \cdot)$ on V.

Solution: $A = {}_S[L]_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ for this map since $L(v_1) = v_2$ and $L(v_2) = -v_1$. We check

$$A^T MA = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \neq M$$

shows that this map is not orthogonal with respect to $(\cdot, \cdot)$ on V.