NAME (Printed): ____

Math 507 Advanced Linear Algebra Fall 2025 Quiz 9 - 10 Feingold

Fill in the blanks. No reasons needed to justify your answers.

NOTATIONS: $\mathbb{R}$ is the real numbers, $\mathbb{C}$ is the complex numbers. For any $A = [a_{ij}] \in \mathbb{C}^m_n$, the complex conjugate of A is $\overline{A} = [\bar{a}_{ij}]$. Each problem is worth 1 point.

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(1) If $A, B \in \mathbb{R}^n_n$ and $(AX) \cdot Y = X \cdot (BY)$ for all $X, Y \in \mathbb{R}^n$, then the **relationship** between A and B is ________________

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(2) For $v_1, \cdots, v_k \in \mathbb{R}^n$, the **most general situation** when you can be sure $||v_1 + \cdots + v_k||^2 = ||v_1||^2 + \cdots + ||v_k||^2$ is when the set $\{v_1, \cdots v_k\}$ is ________________

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(3) The Triangle Inequality in $\mathbb{R}^n$, $||X + Y|| \leq ||X|| + ||Y||$ for any $X, Y \in \mathbb{R}^n$, implies that for any $v_1, \cdots, v_k \in \mathbb{R}^n$ we have $||v_1 + \cdots + v_k|| \leq$ ________________

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(4) For $Z, W \in \mathbb{C}^n$ we have the dot product $Z \cdot W = Z^T \overline{W}$, where $\overline{W}$ is the complex conjugate of W. If $A, B \in \mathbb{C}^n_n$ and $(AZ) \cdot W = Z \cdot (BW)$ for all $Z, W \in \mathbb{C}^n$, then the **relationship** between A and B is ________________

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(5) A matrix $A \in \mathbb{C}^n_n$ is called **unitary** when $\overline{A}^T = A^{-1}$. Using the fact that $\det(\overline{A}) = \overline{\det(A)}$ for any matrix A, we can say that for A unitary, $\det(A) = z = a + b\mathbf{i} \in \mathbb{C}$ must satisfy the following condition on a and b: ________________

Show all work for these problems.

(6) (2 Pts) Show that $M = [m_{ij}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is **positive definite**.

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(7) (3 Pts) The matrix M from (6) defines an inner product on $\mathbb{R}^3$ by the formula $(X, Y) = X^T M Y$. Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$ and let $\theta_{X,Y}$ be the angle between X and Y in the geometry determined by M. Then:

$\cos(\theta_{\mathbf{e}_1,\mathbf{e}_2}) =$ $\qquad$ $\cos(\theta_{\mathbf{e}_2,\mathbf{e}_3}) =$ $\qquad$ and $\qquad$ $\cos(\theta_{\mathbf{e}_1,\mathbf{e}_3}) =$

Fill in the blanks. No reasons needed to justify your answers, but justifications were included in the solutions for your understanding.

NOTATIONS: $\mathbb{R}$ is the real numbers, $\mathbb{C}$ is the complex numbers. For any $A = [a_{ij}] \in \mathbb{C}^m_n$, the complex conjugate of A is $\overline{A} = [\bar{a}_{ij}]$. Each problem is worth 1 point.

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(1) If $A, B \in \mathbb{R}^n_n$ and $(AX) \cdot Y = X \cdot (BY)$ for all $X, Y \in \mathbb{R}^n$, then the **relationship** between A and B is: $\underline{A^T = B}$.
Justification: $(AX) \cdot Y = (AX)^T Y = X^T A^T Y = X \cdot (A^T Y) = X \cdot (BY)$ so $X \cdot (A^T Y - BY) = 0$ is true for all $X, Y \in \mathbb{R}^n$. This being true for all $X \in \mathbb{R}^n$ gives $0^n_1 = A^T Y - BY = (A^T - B)Y$, and that being true for all $Y \in \mathbb{R}^n$ gives $A^T - B = 0^n_n$ so $A^T = B$.

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(2) For $v_1, \cdots, v_k \in \mathbb{R}^n$, the **most general situation** when you can be sure $||v_1 + \cdots + v_k||^2 = ||v_1||^2 + \cdots + ||v_k||^2$ is when the set $\{v_1, \cdots v_k\}$ is orthogonal.
Justification: This is the generalized Pythagorean Theorem.

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(3) The Triangle Inequality in $\mathbb{R}^n$, $||X + Y|| \leq ||X|| + ||Y||$ for any $X, Y \in \mathbb{R}^n$, implies that for any $v_1, \cdots, v_k \in \mathbb{R}^n$ we have $||v_1 + \cdots + v_k|| \leq \underline{||v_1|| + \cdots + ||v_k||}$.
Justification: Follows from the Triangle Inequality by induction on k.

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(4) For $Z, W \in \mathbb{C}^n$ we have the dot product $Z \cdot W = Z^T \overline{W}$, where $\overline{W}$ is the complex conjugate of W. If $A, B \in \mathbb{C}^n_n$ and $(AZ) \cdot W = Z \cdot (BW)$ for all $Z, W \in \mathbb{C}^n$, then the **relationship** between A and B is: $\underline{\overline{A}^T = B}$.
Justification: $(AZ) \cdot W = (AZ)^T \overline{W} = Z^T A^T \overline{W} = Z^T \overline{(\overline{A}^T W)} = Z \cdot (\overline{A}^T W) = Z \cdot (BW)$ true for all $Z, W \in \mathbb{C}^n$. The rest of the argument is as in problem (1).

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(5) A matrix $A \in \mathbb{C}^n_n$ is called **unitary** when $\overline{A}^T = A^{-1}$. Using the fact that $\det(\overline{A}) = \overline{\det(A)}$ for any matrix A, we can say that for A unitary, $\det(A) = z = a + b\mathbf{i} \in \mathbb{C}$ must satisfy the condition $\underline{z\bar{z} = a^2 + b^2 = 1}$.
Justification: $\overline{A}^T = A^{-1}$ means $I_n = A\overline{A}^T$ so $1 = \det(I_n) = \det(A\overline{A}^T) = \det(A)\det(\overline{A}^T) = \det(A)\overline{\det(A)} = z\bar{z}$. Note that for $z = a + b\mathbf{i} \in \mathbb{C}$, $\bar{z} = a - b\mathbf{i}$ so $z\bar{z} = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2$. Then the condition on z is that $a^2 + b^2 = 1$, which is a circle in $\mathbb{C}$, which could be written as $\{z = a + b\mathbf{i} = \cos(\phi) + \mathbf{i}\sin(\phi) \in \mathbb{C} \mid 0 \leq \phi \leq 2\pi\}$.

Show all work for these problems.

(6) (2 Pts) Show that $M = [m_{ij}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is **positive definite**.

Solution: We have $[\, x \quad y \quad z\,] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz =$ $x^2 + (x-y)^2 + (y-z)^2 + z^2 \geq 0$ since it is the sum of real squares. That expression is 0 iff $0 = x = x - y = y - z = z$ iff $x = y = z = 0$ so M is positive definite.

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(7) (3 Pts) The matrix M from (6) defines an inner product on $\mathbb{R}^3$ by the formula $(X, Y) = X^T M Y$. Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$ and let $\theta_{X,Y}$ be the angle between X and Y in the geometry determined by M. Then:

$\cos(\theta_{\mathbf{e}_1,\mathbf{e}_2}) =$ $\qquad$ $\cos(\theta_{\mathbf{e}_2,\mathbf{e}_3}) =$ $\qquad$ and $\qquad$ $\cos(\theta_{\mathbf{e}_1,\mathbf{e}_3}) =$

Solution: Let $M = [m_{ij}]$ from (1). Since $(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i^T M \mathbf{e}_j = m_{ij}$ and

$$\cos(\theta_{\mathbf{e}_i,\mathbf{e}_j}) = \frac{\langle \mathbf{e}_i, \mathbf{e}_j \rangle}{(||\mathbf{e}_i||)(||\mathbf{e}_j||)} = \frac{m_{ij}}{\sqrt{2}\sqrt{2}} = \frac{m_{ij}}{2}$$

we get

$\cos(\theta_{\mathbf{e}_1,\mathbf{e}_2}) = \dfrac{-1}{2}$ $\qquad$ $\cos(\theta_{\mathbf{e}_2,\mathbf{e}_3}) = \dfrac{-1}{2}$ $\qquad$ and $\qquad$ $\cos(\theta_{\mathbf{e}_1,\mathbf{e}_3}) = 0$
