Sophus Lie: Norwegian mathematician, 1842-1899.

<u>Def.</u> Let L be a vector space over field F.
Say L is a Lie algebra when have <u>bilinear</u> map
$[\cdot,\cdot] : L \times L \to L$ satisfying

① $[x,x]=0$, $\forall x \in L$,

② $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$, $\forall x,y,z \in L$.
(Jacobi Identity)

Call $[x,y]$ the "Lie bracket" of x and y.

<u>Note</u>: ① implies $0=[x+y,x+y]=[x,x]+[y,y]+[x,y]+[y,x] = [x,y]+[y,x]$ using bilinearity, so
have ①': $[x,y]=-[y,x]$, $\forall x,y \in L$ (anti-symmetry).

If char$(F) \neq 2$ then ①$' \Rightarrow$ ①.

<u>Note</u>: ② is equivalent to

②$'$: $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$

To interpret the meaning of ②$'$, define "ad".

<u>Def</u>. For L any Lie algebra, $\forall x \in L$, define linear map $ad_x : L \to L$ by $ad_x(y) = [x,y]$, $\forall y \in L$.

<u>Def</u>. For L any Lie algebra, define linear map $ad : L \longrightarrow End(L)$ by $ad(x) = ad_x$, $\forall x \in L$.

<u>Th</u>: ②$'$ says $(ad_x \circ ad_y)(z) - (ad_y \circ ad_x)(z) = ad_{[x,y]}(z)$, $\forall x,y,z \in L$ so $ad_x \circ ad_y - ad_y \circ ad_x = ad_{[x,y]}$ in $End(L)$.

② $'$ also says

$$\mathrm{ad}_x[y,z] = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)].$$

__Def__. Suppose A is an __algebra__ over field F with addition $+$ and product $*$ (need not be assoc.).

Say a linear map $\delta : A \to A$ is a __derivation__ of A when $\delta(x*y) = (\delta(x)*y) + (x*\delta(y))$.

__Note__: ad_x is a derivation of Lie algebra L.

__Th__. Let $(A,+,\cdot)$ be any associative algebra. Define $[x,y] = x\cdot y - y\cdot x$, $\forall x,y \in A$. Then under this "commutator" bilinear operation, $(A,+,[\cdot,\cdot])$ is a Lie algebra.

__Pf__. Exercise. ② requires assoc. of $\cdot$.

Examples: The assoc. algebra F_n^n of $n \times n$ square matrices over F with $\cdot$ as matrix multiplication becomes the Lie algebra $gl(n,F)$ with Lie bracket $[X,Y] = XY - YX$.

For V any vector space over F, the assoc. algebra $End(V) = \{L : V \to V \mid L \text{ is linear}\}$ with composition as product becomes the Lie algebra $gl(V)$ with Lie bracket $[K,L] = K \circ L - L \circ K$.

Def. If L is a Lie algebra and K is a subspace of L, say K is a Lie subalgebra of L when K is itself a Lie algebra under the $[\cdot,\cdot]$ from L. Notation for subalgebra: $K \leq L$.

Th: For subspace K in L, K is a Lie subalg. when K is closed under $[\cdot,\cdot]$, that is, when $[K,K] \subseteq K$, $[x,y] \in K$, $\forall x,y \in K$.

Ex. Let $L = \mathfrak{gl}(n,F)$ and let $K = \mathfrak{sl}(n,F) = \{x \in \mathfrak{gl}(n,F) \mid Tr(x) = 0\}$. We have $Tr([x,y]) = Tr(xy - yx) = Tr(xy) - Tr(yx) = 0$ $\forall x,y \in L$, so $[L,L] \subseteq K$ so $[K,K] \subseteq K$ makes $K \leq L$.

Important Case: $n=2$, $\mathfrak{sl}(2,F)$ has basis $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and brackets $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$.

In $\mathfrak{gl}(n,F)$ define "standard basis" matrices $\{E_{ij} \mid 1 \leq i, j \leq n\}$ where E_{ij} is the $n \times n$ matrix with 1 in row i, column j, 0 in all other entries. So under matrix multiplication

$$E_{ij} E_{kl} = \delta_{jk} E_{il} \quad \text{where } \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

is the "Kronecker delta" function.

Then $[E_{ij}, E_{kl}] = E_{ij} E_{kl} - E_{kl} E_{ij}$

$$= \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

This formula contains "hidden" structure in $\mathfrak{gl}(n,F)$ and its subalgebras, such as $\mathfrak{sl}(n,F)$.

Def. In Lie algebra $L = \mathfrak{sl}(n+1, F)$ define
$h_i = E_{ii} - E_{(i+1)(i+1)}$ for $1 \le i \le n$ and let
$e_i = E_{i(i+1)}$ and $f_i = E_{(i+1)i}$. Let $H = \mathrm{span}\{h_i\}$

Exercise: ① Show that for $1 \le i, j \le n$, we have
$[h_i, h_j] = 0$.

② Compute $[h_i, e_j]$ and $[h_i, f_j]$.

Write the answers in the form
$[h_i, e_j] = \alpha_j(h_i)\, e_j$ for a lin. map $\alpha_j : H \to F$
and $[h_i, f_j] = -\alpha_j(h_i)\, f_j$.

<u>Def</u>. An <u>ideal</u> K in Lie alg. L is a subspace of L such that $[K,L] \subseteq K$. Notation: $K \trianglelefteq L$.

Ideals in Lie algebras are like ideals in rings, can form "quotient" Lie alg. L/K using cosets $\{x+k \mid k \in K\} = x+K$, and brackets (well-defined for $K \trianglelefteq L$)

$$[x+K,\, y+K] = [x,y]+K$$

<u>Def</u>. A Lie alg. <u>homomorphism</u> between two Lie algebras is a linear map $\phi: L \longrightarrow K$ s.t.

$$\phi[x,y] = [\phi(x), \phi(y)] \quad \forall x,y \in L.$$

Lie bracket in L Lie bracket in K

<u>Ex</u>: $ad: L \to gl(L)$ since $ad_{[x,y]} = [ad_x, ad_y]$.

Def. Let L be a Lie algebra, and let V be a vector space. Say lin. map $\phi : L \to \mathrm{End}(V)$ is a **representation** of L on V when $\forall x, y \in L$,

$$\phi[x,y] = \phi(x) \circ \phi(y) - \phi(y) \circ \phi(x),$$

that is, $\phi : L \to \mathfrak{gl}(V)$ is a Lie alg. hom.

$$\phi[x,y] = [\phi(x), \phi(y)] \quad \text{(commutator)}.$$

Exercise: ① For $\phi : L \to K$ a hom. of Lie algs. show $\mathrm{Ker}(\phi) \trianglelefteq L$.

② For $\mathrm{ad} : L \to \mathfrak{gl}(L)$ what can you say about $\mathrm{Ker}(\mathrm{ad})$?

Def. For Lie alg. L, say L is <u>abelian</u> when $[x,y]=0$, $\forall x,y \in L$, so $[L,L]=\{0\}$.

Def. The "derived" ideal of L is $L' = [L,L] = \operatorname{span}\{[x,y] \mid x,y \in L\}$.

Def. The "center" of L is
$Z(L) = \{x \in L \mid [x,y]=0, \forall y \in L\}$. $\boxed{\text{Prove } Z(L) \trianglelefteq L}$

Def. Say two Lie algebras L and H are <u>isomorphic</u> when $\exists\, \phi : L \to H$ a Lie alg. isomorphism, that is, a bijective hom.

Def. Say L is <u>simple</u> when its only ideals are $\{0\}$ and L, and $[L,L] \neq \{0\}$.

<u>Note:</u> $\dim(L)=1 \Rightarrow [L,L]=\{0\}$.