We know $ad_s$ is the s.s. part of $ad_x$ (see p. 97)
so it is a poly in $ad_x$ with 0 constant term.
Since $r(0)=0$, $r(T)$ has 0 const. term, so $ad_y =$
$r(ad_s)$ is a poly. in $ad_x$ with 0 const. term.
We are assuming $x \in M = \{x \in gl(V) \mid ad_x(B) \subset A\}$
so $ad_y(B) \subset A$, that is, $y \in M$. We are also given
$Tr(xy)=0$ since $y \in M$. The matrix rep'ing $xy$
w.r.t. basis $\mathcal{S}$ is $diag(a_1 f(a_1), \dots, a_m f(a_m))$ so

$$Tr(xy) = \sum_{i=1}^{m} a_i f(a_i) = 0.$$

This sum is a lin. comb. in $E$ with coeffs. $f(a_i) \in Q$ so if we apply the map $f \in E^*$ we get $\sum_{i=1}^{m} f(a_i)^2 = 0$. Since $f(a_i) \in Q$, we must have all $f(a_i)=0$, so $f$ is the zero function in $E^*$. $f \in E^*$ was arbitrary so $E^*=0$. □

<u>Lemma</u>: $\forall x, y, z \in End(V)$, $\dim(V) < \infty$, we have
$$Tr([x,y]z) = Tr(x[y,z]).$$

<u>Pf</u>. $[x,y]z = (xy - yx)z = xyz - yxz$ and
$x[y,z] = x(yz - zy) = xyz - xzy$. $Tr$ is linear and $Tr(y(xz)) = Tr((xz)y)$. $\square$

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<u>Th</u>. (Cartan's Criterion) Let $L \leq gl(V)$, $\dim(V) < \infty$. Suppose $Tr(xy) = 0$, $\forall x \in [L,L]$ and $\forall y \in L$. Then $L$ is solvable.

<u>Pf</u>. By the notes on p. 100, it is enough to show $[L,L]$ is nilp., or that $\forall x \in [L,L]$, $ad_x : [L,L] \to [L,L]$ is nilp. We will apply the last Lemma with $V$ as given, $A = [L,L]$, $B = L$ so $M = \{x \in gl(V) \mid [x, L] \subset [L,L]\}$ and clearly $L \subset M$. Given $Tr(xy) = 0$ for $x \in [L,L]$, $y \in L$,

But the Lemma requires knowing that $Tr(xy)=0 \;\forall x \in [L,L],\, y \in M$, in order to conclude $x$ is nilp. $L \subset M$ but $M$ could be strictly bigger.
Using any $x, y \in L$, $[L,L]$ is spanned by elements $[x,y]$. Let $z \in M$ and use last Lemma to get
$$Tr([x,y]z) = Tr(x[y,z]) = Tr([y,z]x).$$
By def. of $M$, $[y,z] \in [L,L]$ and $x \in L$ so our assumption gives $Tr([y,z]x) = 0$ as required. □

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<u>Cor</u>. Let $L$ be a Lie alg. s.t. $Tr(ad_x \circ ad_y) = 0$ $\forall x \in [L,L],\, y \in L$. Then $L$ is solvable.

<u>Pf</u>. Apply Theorem to $ad(L) = \{ad_x \in gl(L) \mid x \in L\}$ get hom. image $ad(L)$ is solv. $Ker(ad) = Z(L)$ is solv. so $L$ is solv. by previous Thm. (Notes p.57) □

**Killing Form:** Let $L$ be a Lie algebra.

**Def.** For any $x,y \in L$, let $K(x,y) = Tr(ad_x \circ ad_y)$. Then $K : L \times L \to F$ is a symmetric bilinear form on $L$ called the **Killing form**. Also, $\forall\, x,y,z \in L$, we have

$$K([x,y],z) = K(x,[y,z]) \quad \text{(associativity of } K\text{)}$$

since $Tr([x,y]z) = Tr(x[y,z]) \ \forall\, x,y,z \in End(V)$.

**Lemma:** Let $I \trianglelefteq L$, $K : L \times L \to F$ as above, $K_I : I \times I \to F$ the Killing form for $I$ as a Lie algebra. Then

$$K_I = K|_{I \times I}.$$

**Pf.** Note that for any subspace $W \leq V$ and any $\phi \in End(V)$ s.t. $\phi(V) \subseteq W$, we have $Tr(\phi) = Tr(\phi|_W)$. To see this, let $S = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ be a basis of $V$ where $T = \{v_1, \dots, v_k\}$ is a basis of $W$. Then the matrix representing $\phi$ w.r.t. $S$ has the form

$${}_S[\phi]_S = \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right]_{n\times n}$$ for $A \in F_k^k$ since $\phi(v_i) \in W = \langle T \rangle$.

If $x, y \in I$, $ad_x \circ ad_y : L \to I$ so $K(x,y) = Tr(ad_x \circ ad_y) = Tr((ad_x \circ ad_y)|_I) = Tr(ad_x|_I \circ ad_y|_I) = K_I(x,y)$. $\square$

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<u>Def</u>. For any symmetric bilinear form $\beta : L \times L \to F$ call it <u>nondegenerate</u> if $Rad(\beta) = 0$ where $S = Rad(\beta) = \{x \in L \mid \beta(x,y) = 0, \forall y \in L\}$.

$\beta$ associative $\Rightarrow Rad(\beta) \trianglelefteq L$ since $\forall x \in S, \forall y, z \in L$, $\beta([x,y], z) = \beta(x, [y,z]) = 0$ so $[x,y] \in S$.

For any basis $T = \{x_1, \ldots, x_n\}$ of $L$, the matrix of $\beta$ w.r.t. $T$ is ${}_T M_T = [\beta(x_i, x_j)] \in F_n^n$ and

$\forall\, x, y \in L$, if $[x]_T$ and $[y]_T$ are the coordinates w.r.t. $T$, then

$$\beta(x,y) = \underbrace{[x]_T^{Tr}}_{(1\times n)}\ \underbrace{{}_T M_T}_{(n\times n)}\ \underbrace{[y]_T}_{(n\times 1)}$$

(Tr means transpose)

so $\beta$ is symm. iff ${}_T M_T$ is symm, and $\beta$ is nondeg. iff $\operatorname{rank}({}_T M_T) = n$ iff $\det({}_T M_T) \neq 0$.

<u>EX.</u> For $L = \mathfrak{sl}(2,F)$ with std. basis $\{e, f, h\} = T$

$$ad_h : \begin{cases} e \to 2e \\ f \to -2f \\ h \to 0 \end{cases} \quad \text{so} \quad {}_T[ad_h]_T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = H$$

$$ad_e : \begin{cases} e \to 0 \\ f \to h \\ h \to -2e \end{cases} \quad \text{so} \quad {}_T[ad_e]_T = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = E$$

$$ad_f : \begin{cases} e \to -h \\ f \to 0 \\ h \to 2f \end{cases} \quad \text{so} \quad {}_T[ad_f]_T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix} = F$$

differs from order in Humphreys' book.

For convenient notation, let $X_1 = E$, $X_2 = F$, $X_3 = H$ 109

so $[K(X_i, X_j)] = [\mathrm{Tr}(X_i X_j)] = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} = M$ since

$$E^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X_1^2 \text{ has trace } 0,$$

$$EF = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = X_1 X_2 \text{ has trace } 4$$

$$EH = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} = X_1 X_3 \text{ has trace } 0$$

$$F^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X_2^2 \text{ has trace } 0$$

$$FH = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = X_2 X_3 \text{ has trace } 0$$

$$H^2 = \begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 4 & & \\ & 4 & \\ & & 0 \end{bmatrix} = X_3^2 \text{ has trace } 8$$

and $\det(M) = -128 \neq 0$

so $K$ is nondegenerate.