More examples of Lie algebras:

Let V be a v.s. over F and let $\langle \cdot, \cdot \rangle : V \times V \to F$ be a bilinear form on V. Let

$$\mathcal{L} = \{ L : V \to V \mid \langle L(v_1), v_2 \rangle = -\langle v_1, L(v_2) \rangle \ \forall v_1, v_2 \in V \}$$

Then $\mathcal{L} \leq \mathfrak{gl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

<u>Pf</u>. It is easy to check that $\mathcal{L}$ is a subspace of $\mathfrak{gl}(V)$, so all we need to check is closure under commutator. Suppose $L_1, L_2 \in \mathcal{L}$, then

$$\langle [L_1, L_2](v_1), v_2 \rangle = \langle (L_1 \circ L_2 - L_2 \circ L_1)(v_1), v_2 \rangle$$

$$= \langle L_1(L_2(v_1)), v_2 \rangle - \langle L_2(L_1(v_1)), v_2 \rangle$$

$$= -\langle L_2(v_1), L_1(v_2) \rangle + \langle L_1(v_1), L_2(v_2) \rangle =$$

$$= \langle v_1, L_2(L_1(v_2)) \rangle - \langle v_1, L_1(L_2(v_2)) \rangle = -\langle v_1, [L_1, L_2](v_2) \rangle.$$

Let's view $\mathcal{L}$ from the matrix point of view.

Let V have basis $S = \{v_1, \dots, v_n\}$ and let ${}_S M_S = [\langle v_i, v_j \rangle]$ be the $n \times n$ matrix of the bilinear form w.r.t. S. Using notation $[v]_S = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n$ when $v = \sum_{j=1}^{n} a_j v_j$ for the coordinates of $v \in V$ w.r.t. S, we have

$$\underset{\forall v, w \in V}{\langle v, w \rangle} = \underset{1 \times n}{[v]_S^{Tr}} \; \underset{n \times n}{{}_S M_S} \; \underset{n \times 1}{[w]_S}$$

if a 1×1 matrix is just a scalar.

<u>Note</u>: If $[w]_S = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ then both sides of the above equation equal

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \langle v_i, v_j \rangle .$$

For $L: V \to V$ we use the notation $A = {}_S[L]_S \in F^n_n$ for the matrix representing L with respect to S, so $\forall v \in V$ we have

$${}_S[L]_S [v]_S = [L(v)]_S .$$

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\ ]_S}"] & V \arrow[d, "{[\ ]_S}"] \\ F^n \arrow[r, "{{}_S[L]_S}", "n \times n"'] & F^n \end{tikzcd}$$

Then $L \in \mathcal{L}$ means, $\forall v, w \in V$,

$$\langle L(v), w \rangle = -\langle v, L(w) \rangle$$ iff

$$[L(v)]_S^{Tr} \, {}_SM_S [w]_S = -[v]_S^{Tr} \, {}_SM_S [L(w)]_S$$ iff

$$(A[v]_S)^{Tr} \, {}_SM_S [w]_S = -[v]_S^{Tr} \, {}_SM_S (A[w]_S)$$ iff

$$[v]_S^{Tr} (A^{Tr} \, {}_SM_S) [w]_S = -[v]_S^{Tr} ({}_SM_S A) [w]_S$$

Since this is true $\forall v, w \in V$, it is true $\forall X = [v]_S,\ Y = [w]_S \in F^n$, that is,

$$X^{Tr}\left(A^{Tr}\,{}_SM_S\right)Y = -X^{Tr}\left({}_SM_S\,A\right)Y.$$

Using $v = v_i \in S$, $w = v_j \in S$, we get $X = e_i$, $Y = e_j$ standard basis vectors in F^n and $e_i^{Tr}\, B\, e_j = b_{ij}$ for any $B = [b_{ij}] \in F^n_n$, so the $(i,j)^{th}$ entries of $A^{Tr}\,{}_SM_S$ and $-{}_SM_S\,A$ are equal for all $1 \le i, j \le n$, so

$A^{Tr}\,{}_SM_S = -{}_SM_S\,A$ is the condition on $A = {}_S[L]_S$ equivalent to $L \in \mathcal{L}$.

<u>Th</u>: For any $n\times n$ matrix $M \in F^n_n$, let $\mathcal{L}(M) = \{A \in F^n_n \mid A^{Tr} M = -MA\}$. Then $\mathcal{L}(M)$ is a Lie subalgebra of $\mathfrak{gl}(n,F)$.

The most important special cases are:

① $\langle v, w\rangle = \langle w, v\rangle$ symmetric bilinear form (non-degenerate)

② $\langle v, w\rangle = -\langle w, v\rangle$ antisymmetric bil. form (non-deg.) $\Rightarrow$ $\dim(V)$ is even

corresponding to

① $M = M^{Tr}$ sym. non-deg. matrix

② $M = -M^{Tr}$ antisym non-deg. matrix (n even)

Call the corresponding Lie algebra "orthogonal" in case ①, "symplectic" in case ②.

<u>Ex</u>: $M = I_n = M^{Tr}$ gives

$\mathcal{L}(M) = \mathcal{L}(I_n) = \{A \in F^n_n \mid A^{Tr} = -A\}$.

In textbook (Humphreys), for a more uniform presentation, he used M in block form:

① $M = \left[\begin{array}{c|c} 0 & I_\ell \\ \hline I_\ell & 0 \end{array}\right] \in F^{2\ell}_{2\ell}$, $n = 2\ell$ even: $\boxed{o(2\ell, F)}$ "Type D_ℓ"

①' $M = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{array}\right] \in F^{2\ell+1}_{2\ell+1}$, $n = 2\ell+1$ odd: $\boxed{o(2\ell+1, F)}$, "Type B_ℓ"

② $M = \left[\begin{array}{c|c} 0 & I_\ell \\ \hline -I_\ell & 0 \end{array}\right] \in F^{2\ell}_{2\ell}$, $n = 2\ell$: $\boxed{sp(2\ell, F)}$, "Type C_ℓ"

while $s\ell(\ell+1, F)$ is called "Type A_ℓ".

Using block forms for $A \in \mathcal{L}$ in each case [1]
we can get precise conditions on each block
from $A^{Tr} M = -MA$ giving an explicit basis
for each type of "classical matrix Lie alg."
in these four series of Lie algebras.

Get $\dim(\mathfrak{o}(2\ell, F)) = 2\ell^2 - \ell$

$\dim(\mathfrak{o}(2\ell+1, F)) = 2\ell^2 + \ell$

$\dim(\mathfrak{sp}(2\ell, F)) = 2\ell^2 + \ell$

$\dim(\mathfrak{sl}(\ell+1, F)) = (\ell+1)^2 - 1$.

Other important Lie subalgebras of $\mathfrak{gl}(n,F)$:

$\mathfrak{t}(n,F) = \{A = [a_{ij}] \in F^n_n \mid a_{ij} = 0 \text{ if } i > j\}$ (upper Δ)

$\mathfrak{n}(n,F) = \{A = [a_{ij}] \in F^n_n \mid a_{ij} = 0 \text{ if } i \geq j\}$ (strictly upper Δ)

$\mathfrak{d}(n,F) = \{ \cdots\cdots \mid a_{ij} = 0 \text{ if } i \neq j\}$ (diagonal)

<u>Facts:</u> $t(n,F) = d(n,F) \oplus n(n,F)$ is a direct sum as vector spaces. Let $t = t(n,F)$.

$[d(n,F), d(n,F)] = 0$ so $d = d(n,F)$ is abelian.

$[d,t] \leq t$ and actually, $[d,t] = n$.

For $0 \leq k$ let $t_k = \bigoplus_{i=1}^{n} F E_{i(i+k)}$ so

$$t_0 = \left\{ \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} \right\} = d, \quad t_1 = \left\{ \begin{bmatrix} 0 & a_{12} & & & 0 \\ & 0 & a_{23} & & \\ & & 0 & \ddots & \\ & & & \ddots & a_{(n-1)n} \\ 0 & & & & 0 \end{bmatrix} \right\}$$

$$t_2 = \left\{ \begin{bmatrix} 0 & 0 & a_{13} & & 0 \\ & 0 & 0 & a_{24} & \\ & & \ddots & 0 & \ddots \\ & & & \ddots & \vdots \\ 0 & & & & 0 \end{bmatrix} \right\}, \ldots t_{n-1} = \left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 & a_{1n} \\ & 0 & & & 0 \\ & & \ddots & & \vdots \\ 0 & & & \ddots & 0 \end{bmatrix} \right\}$$

Check: $[t_i, t_j] \leq t_{i+j}$ as <u>subspaces</u>, since

$$[E_{r(r+i)}, E_{s(s+j)}] = \delta_{(r+i),s} E_{r(s+j)} - \delta_{(s+j),r} E_{s(r+i)}$$

$$= \delta_{(r+i),s} E_{r(r+i+j)} - \delta_{(s+j),r} E_{s(s+i+j)}.$$

For $k \geq n$, $t_k = \{0^n_n\}$.

<u>Def.</u> A Lie algebra L is "$\mathbb{Z}$-graded" when $\forall k \in \mathbb{Z}$, there is a subspace L_k of L s.t. $[L_k, L_m] \subseteq L_{k+m}$ $\forall k, m \in \mathbb{Z}$, and $L = \bigoplus_{k \in \mathbb{Z}} L_k$ is a direct sum of subspaces.

Can have some $L_k = \{0\}$. Can be generalized.

Examples of representations:

For $L = gl(n, F)$, define a rep'n of L on $V = F^n$ (column vectors) by left matrix multiplication; $\phi : L \to End(F^n)$ is $\phi(A)(X) = AX$, $\forall A \in F_n^n$, $X \in F^n$. Then

$$\phi([A,B])(X) = (A \circ B - B \circ A)X = A(BX) - B(AX)$$
$$= \phi(A)(\phi(B)(X)) - \phi(B)(\phi(A)(X))$$
$$= (\phi(A)\,\phi(B) - \phi(B)\,\phi(A))X = (AB - BA)X$$
$$= [\phi(A), \phi(B)](X)$$

so ϕ is a Lie alg. rep'n.

If $L \leq gl(n,F)$ is any Lie subalgebra [2] of $gl(n,F)$, we get a rep'n of L on F^n by the same "action", left matrix mult.

<u>Explicit case</u>: $L = sl(2,F)$ rep'd on F^2.

Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_1, \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_2 \right\}$ be std. basis of F^2.

L has basis: $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

<u>Check:</u> $e v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $f v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_2$, $h v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_1$

and $e v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_1$, $f v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $h v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -v_2$

so action of h on $V = F^2$ is diagonal with e-values 1 (on Fv_1), -1 (on Fv_2).

Example: Let $V = \mathbb{F}[x,y]$ v.s. over $\mathbb{F}$ of polynomials in two variables. Define operators

$E = x\partial_y$, $F = y\partial_x$, $H = x\partial_x - y\partial_y$ where ∂_x and ∂_y are partial derivatives w.r.t. x and y, respectively. For monomials, $x^i y^j$, we have:

$$(x\partial_y)\, x^i y^j = j\, x^{i+1} y^{j-1},$$
$$(y\partial_x)\, x^i y^j = i\, x^{i-1} y^{j+1}$$

so

$$[E,F]\, x^i y^j = \big((x\partial_y)\circ(y\partial_x) - (y\partial_x)\circ(x\partial_y)\big)\, x^i y^j$$
$$= i(j+1)\, x^i y^j - j(i+1)\, x^i y^j = (i-j)\, x^i y^j$$
$$= (x\partial_x - y\partial_y)\, x^i y^j = H\, x^i y^j.$$

So as linear operators on V, we have $[E,F] = H$.

Exercise: Using the formulas above for the actions of these linear operators on V, check that $[H,E]=2E$ and $[H,F]=-2F$.

Hint: Apply to monomial $x^i y^j$.

Th. The linear map $\phi : \mathfrak{sl}(2,\mathbb{F}) \to End(\mathbb{F}[x,y])$ such that $\phi(e)=E=x\partial_y$, $\phi(f)=F=y\partial_x$ and $\phi(h)=H=x\partial_x - y\partial_y$, is a Lie alg. rep'n.

Def. For $\phi : L \to End(V)$ a rep'n, we say V is an "L-module" where "L acts on V" by $x\cdot v = \phi(x)(v)$, $\forall x \in L$, $\forall v \in V$. So $\cdot : L \times V \to V$ is Just an equivalent terminology with simpler notation: $[x,y]\cdot v = x\cdot(y\cdot v) - y\cdot(x\cdot v)$.

<u>Def.</u> Let V be an L-module and $W \leq V$ a subspace. Say W is an L-sub-module of V when $\forall x \in L$, $\forall w \in W$, $x \cdot w \in W$, that is, W is itself an L-module. This is the same as saying that rep'n $\phi : L \to \mathrm{End}(V)$ has a restriction $\phi_W : L \to \mathrm{End}(W)$ by $\phi_W(x) = \phi(x)\big|_W : W \to W$.

<u>Example:</u> In the previous example where $L = \mathfrak{sl}(2, F)$ and $V = \mathbb{F}[X, Y]$, for $0 \leq m \in \mathbb{Z}$ let $V_m = \mathbb{F}\text{-span}\{X^m, X^{m-1}Y, X^{m-2}Y^2, \ldots, XY^{m-1}, Y^m\}$ $= \mathbb{F}\text{-span}\{X^i Y^j \mid i+j=m\}$ be the subspace of all polynomials in V of homogeneous degree m.

Th: For each $0 \le m \in \mathbb{Z}$ the subspace V_m in V is an L-submodule with $\dim(V_m) = m+1$. 25

Pf. The formulas for the action of E, F, H on monomials $X^i Y^j$ show that the homog. degree $i+j$ is preserved by their action. $\square$

Note: The action of H on $X^i Y^j$ is mult by $i-j$ so H is diagonal with eigenvalues:

$m, m-2, m-4, \ldots, -(m-2), -m$

and these match the "natural" rep'n on $\mathbb{F}^2$ for $m=1$.

The "trivial" rep'n for $m=0$ is 1-dim'l with all operators acting as zero.

Example: Let $V = F[X]$, and let $p = \partial_X$ be differentiation w.r.t. X, $q = X$ be mult. by X.

Then $[p,q]\, X^i = (\partial_X X - X \partial_X) X^i = (i+1)X^i - i X^i$
$= 1 X^i$ so $[p,q] = I$ is the identity operator on V.

of course, $[I,p] = 0 = [I,q]$ so the three operators p, q, I span a 3-dim'l Lie alg. inside $gl(V)$, called a Heisenberg Lie alg., $\underline{h}$

Note: $[\underline{h},\underline{h}] = \mathbb{F} I = \underline{h}'$ so $[\underline{h}',\underline{h}'] = 0$ and $[\underline{h},[\underline{h},\underline{h}]] = 0$. $\underline{h}$ is used in physics as a model of position, momentum, uncertainty principal.