<u>Note</u>: Define $\beta(x,y) = Tr(xy)$ for $x, y \in sl(2,F)$ as 2×2 matrices, giving a symm. bil. assoc. form on $L = sl(2,F)$ using its "natural" rep'n on F^2 as 2×2 matrices. With same basis: $x_1 = e$, $x_2 = f$, $x_3 = h$,

$$Tr(x_1^2) = Tr\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = Tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = Tr(e^2)$$

$$Tr(x_1 x_2) = Tr\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = Tr\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 = Tr(ef)$$

$$Tr(x_1 x_3) = Tr\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = Tr\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = 0 = Tr(eh)$$

$$Tr(x_2^2) = Tr\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = Tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = Tr(f^2)$$

$$Tr(x_2 x_3) = Tr\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = Tr\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 = Tr(fh)$$

$$Tr(x_3^2) = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = Tr\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 = Tr(h^2)$$

So $[\beta(x_i, x_j)] = [Tr(x_i x_j)] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4}[K(x_i, x_j)]$.

Recall: L is semisimple (s.s.) when $\mathrm{Rad}(L) = 0$ III
(max. solvable ideal in L is trivial). Let $S = \mathrm{Rad}(K)$

L is s.s. iff L has no nonzero abelian ideals
because any such ideal is solv. so is in $\mathrm{Rad}(L)$,
and if L is not s.s. then its nonzero $\mathrm{Rad}(L)$ is
a solv. ideal whose derived series has a last
nonzero term $L^{(i-1)} \trianglelefteq L$ which is abelian since
$0 = L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$.

<u>Th</u>: Lie alg. L is s.s. iff K is nondeg.

<u>Pf</u>. ($\Rightarrow$) Suppose $\mathrm{Rad}(L) = 0$. By def. of $S = \mathrm{Rad}(K)$,
$\mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = 0$, $\forall x \in S$, $y \in L$, so true $\forall y \in [S,S]$.
By Cartan's Criterion, $\{\mathrm{ad}_s : L \to L \mid s \in S\} = \mathrm{ad}(S) \subseteq \mathfrak{gl}(L)$
is solvable, so S is solv. ideal in L, so $S \subseteq \mathrm{Rad}(L) = 0$.

($\Leftarrow$) Suppose $S=0$. Show every <u>abelian</u> $I \trianglelefteq L$ is inside S, so is trivial, giving L is s.s.

Let $x \in I$, $y \in L$, so $L \xrightarrow{ad_y} L \xrightarrow{ad_x} I$ so

$(ad_x \circ ad_y): L \to I$ and $(ad_x \circ ad_y)^2(L) \subseteq (ad_x \circ ad_y) I$

$\subseteq [x,[y,I]] \subseteq [I,I] = 0$. This says $(ad_x \circ ad_y)$ is nilp. so $Tr(ad_x \circ ad_y) = 0$ (sum of e-values, all 0)

so $K(x,y)=0$ for $x \in I$, $y \in L$ says $I \subseteq S = 0$. $\square$

<u>Note</u>: First part of proof showed that $S = Rad(K) \subseteq Rad(L)$ in general.

<u>Def.</u> L with ideals $I_1, \dots, I_t \trianglelefteq L$ is a <u>direct sum</u> $L = I_1 \oplus \cdots \oplus I_t$ when it is a direct sum of the $I_1, \dots, I_t$ considered as subspaces.

In that case, $[I_i, I_j] \subseteq I_i \cap I_j = 0$ for $i \neq j$, so L can be viewed as an "external" direct sum of separate Lie algebras $\{I_i \mid 1 \le i \le t\}$ with brackets 0 between different summands.

Ex: $sl(2,F) \oplus sl(3,F) = \{X+Y \mid X \in sl(2,F), Y \in sl(3,F)\}$ with $[X_1+Y_1, X_2+Y_2] = [X_1, X_2] + [Y_1, Y_2]$. Could also use component notation:

For Lie algebras $L_1, \dots, L_t$ let $L_1 \oplus \cdots \oplus L_t = \{(X_1, \dots, X_t) \mid X_i \in L_i, 1 \le i \le t\}$ with componentwise operations $+, \cdot, [\ ,\ ]$. Then, each L_i embeds in L as its component, and $[L_i, L_j] = 0$ for $i \neq j$.

Th. Let L be s.s. Then $\exists$ ideals $L_1, \dots, L_t \trianglelefteq L$ each a simple Lie algebra s.t. $L = L_1 \oplus \cdots \oplus L_t$. Every simple ideal of L is one of these L_i, and $K_{L_i} = K\big|_{L_i \times L_i}$ for the Killing form K_{L_i} on L_i and K on L. 114

Pf. Step ①: $\forall I \trianglelefteq L$ let $I^\perp = \{x \in L \mid K(x,y) = 0, \forall y \in I\}$. By assoc. of K, we see that $I^\perp \trianglelefteq L$. Apply Cartan's Crit. to Lie alg. I to get $I \cap I^\perp \trianglelefteq L$ is solv. so $I \cap I^\perp = 0$. Details: $ad(I) = \{ad_x \in gl(L) \mid x \in I\} \leq gl(L)$ and let $J = I \cap I^\perp$. C.C. gives J solvable if $Tr(ad_{[x,y]} \circ ad_z) = 0$, $\forall x,y,z \in J$. But that is

$$K([x,y],z) = K(x,[y,z]) = 0 \text{ since } x \in I^\perp, [y,z] \in I.$$

K nondeg. $\Rightarrow \dim I + \dim I^\perp = \dim(L)$ so $L = I \oplus I^\perp$.

Step ②: By ind. on $\dim(L)$. If L has no proper §15 nonzero ideal then L is simple and $L=L_1$ is the decomposition. Otherwise, let L_1 be a minimal nonzero ideal, and write $L=L_1 \oplus L_1^{\perp}$. Any ideal of L_1 is an ideal of L and any ideal of $L_1^{\perp}$ is an ideal of L. So $\mathrm{Rad}(L_1) \trianglelefteq L$ s.s. $\Rightarrow L_1$ is ss. and also for $L_1^{\perp}$ s.s. (actually L_1 is simple by minimality).

By ind. $L_1^{\perp} = L_2 \oplus \cdots \oplus L_t$ has a direct sum decomp. into simple ideals which are ideals of L.

Uniqueness: For simple $I \trianglelefteq L$ have $[I,L] \trianglelefteq L$ and $[I,L] \neq 0$ since otherwise $I \subseteq Z(L)=0$.

So $0 \neq [I,L] \trianglelefteq I$ with I simple gives $[I,L]=I$.

But $[I,L]=[I,L_1] \oplus [I,L_2] \oplus \cdots \oplus [I,L_t] = I$ must be just one summand, say $[I,L_j]=I \subseteq L_j$ so $I=L_j$. □

Cor: If L is s.s. then $L = [L, L]$, all ideals
and homomorphic images of L are s.s. Each ideal of L is a sum of certain simple ideals of L.

Inner Derivations:

Recall $\forall x \in L, \forall \delta \in Der(L)$, $[\delta, ad_x] = ad_{\delta(x)}$ which implies $ad(L) = \{ad_x \in gl(L) \mid x \in L\} \trianglelefteq Der(L)$.

Th. If L is s.s. then $ad(L) = Der(L)$, that is, $\forall \delta \in Der(L)$, $\delta \in ad(L)$.

Pf. L s.s. gives $Z(L) = 0$ so $ad : L \to ad(L)$ is an isomorphism, and writing $M = ad(L)$, get that the Killing form K_M is non-deg. Letting $D = Der(L)$, $[D, M] \subseteq M$ so $K_M = K_D|_{M \times M}$.

Let $I = M^{\perp} = \{\delta \in D \mid K_D(\delta, y) = 0, \forall y \in M\}$ subspace of D, then K_M non-deg. implies $I \cap M = 0$. But $I \trianglelefteq D$ and $M \trianglelefteq D$ so $[I, M] \subseteq I \cap M = 0$.
$\forall \delta \in I, x \in L$, $ad_{\delta(x)} = [\delta, ad_x] \in [I, M] = 0$, so ad injective implies $\delta(x) = 0$, so $\delta = 0$ and $M^{\perp} = 0$.
$D = M \oplus M^{\perp} = M$ so $Der(L) = ad(L)$. $\square$

Abstract Jordan Decomp. for L s.s.:

Recall: For any F-alg. $(\mathcal{a}, *)$ fin. dim'l, $Der(\mathcal{a})$ contains semisimple and nilp. parts of its elts.
Since for L s.s., $Der(L) = ad(L) \cong L$, $\forall x \in L$ $\exists! s, n \in L$ s.t. $ad_x = ad_s + ad_n$ is J-decomp. in $End(L)$. Also get $x = s + n$ and $[s, n] = 0$, and s is ad-s.s. and n is ad-nilp. $s = x_s$ and $n = x_n$ are s.s. and nilp. parts of x.