But what if L is a linear (matrix) Lie algebra where $\forall X \in L$, X_s and X_n already have a definition? We'll see later (Humphreys, p. 29-30) that the two definitions of Jordan decomp. agree.

But for now, can check it when $L = \mathfrak{sl}(V)$ for $\dim(V) < \infty$, as follows. Write the usual J-decomp $X = X_s + X_n$ in $End(V)$, $\forall X \in L$. Then $Tr(X_n) = 0$ since X_n is nilp. so $X_n \in L$ so $X_s \in L$ since $Tr(X_s) = Tr(X) - Tr(X_n) = 0 - 0 = 0$. Also,

$ad_{X_s} : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is s.s. (p. 95 of these notes)

so $ad_{X_s} : \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ is s.s. (restriction of s.s. is s.s.)

and similarly, ad_{X_n} is nilp. operator on $\mathfrak{sl}(V) = L$.

But $[ad_{X_s}, ad_{X_n}] = ad_{[X_s, X_n]} = 0$ on L, so uniqueness of abstract J. decomp. in L says it is $X = X_s + X_n$.

Recall (page 23 of these notes) the def'n
of a Lie algebra __module__:
Vector space V with an action of L on V,
$\cdot : L \times V \to V$ denoted $(x, v) \mapsto x \cdot v \in V$, is an
L-module when $\forall x, y \in L$, $\forall v, w \in V$, $\forall a, b \in F$,

(M1) $(ax+by)\cdot v = a(x \cdot v) + b(y \cdot v)$

(M2) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$

(M3) $[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

This is equivalent to $\phi : L \to gl(V)$ being a
rep'n of L on V where $x \cdot v = \phi(x)(v)$.

__Def__. An L-mod. hom. is a lin. map $\phi : V \to W$
between L-modules V and W such that $\forall x \in L$,
$$\phi(x \cdot v) = x \cdot \phi(v), \quad \forall v \in V.$$

Def: For V an L-mod., $W \leq V$ subspace, call W an L-submod. of V when W is itself an L-mod. under the given action of L on V.

Note: If $\phi : V \to W$ is an L-mod. hom. (map) then $\mathrm{Ker}(\phi) \leq V$ is an L-submod.

Just need to check closure under the L action.

Expect usual hom. theorems to work, e.g.,

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\pi"] & W \\ V/\mathrm{Ker}(\phi) \arrow[ur, "\bar{\phi}"'] & \end{tikzcd}$$

$\bar{\phi} \circ \pi = \phi$ exists and $\bar{\phi}$ is inj.

ϕ surj $\Rightarrow$ $V/\mathrm{Ker}(\phi) \cong W$.

If $\phi : V \to W$ is bijective (v.s. isom.) call ϕ an L-mod. isom. and say V and W give equivalent rep'ns.

Def. Say L-mod. V is irreducible when the only two submodules are V and $\{0\}$. So $\dim(V)=0$ is not called irred. The trivial L-module V is the 1-dim'l v.s. s.t. $\forall x \in L$, $\forall v \in V$, $x \cdot v = 0$. A trivial action of L on any V is when $x \cdot v = 0$, $\forall x \in L$, $\forall v \in V$.

Def. Say L-mod. V is completely reducible when V is a direct sum of irred. L-submods. That is equivalent to: $\forall W \leq V$ submod, $\exists W' \leq V$ submod. s.t. $V = W \oplus W'$.

Def: For V, W any L-mods, the direct v.s. sum $V \oplus W = \{(v, w) \mid v \in V, w \in W\}$ is an L-mod. under $x \cdot (v, w) = (x \cdot v, x \cdot w)$ componentwise action.

For any L-mod. V and its associated rep'n $\phi: L \to gl(V)$, we have $\phi(L) = \{\phi(x) \in End(V) \mid x \in L\}$ and that subspace of $End(V)$ generates an associative subalgebra of $End(V)$ to which we may add I_V. For any subsp. $W \leq V$, W is invar. under the L action (is an L-submod. of V) iff the W is invariant under the action of the assoc. subalg. of $End(V)$ generated by $\phi(L)$. Results about assoc. alg. modules can be applied.

<u>Schur's Lemma</u>: Let $\phi: L \to gl(V)$ be irred. If $y \in End(V)$ and $[y, \phi(x)] = 0$, $\forall x \in L$, then $y = \alpha I_V$, for some $\alpha \in F$, is a scalar operator on V.

<u>Pf</u>. Over alg. closed field F, y has at least one e.vector in V, say $y(v_0) = \alpha v_0$ for some e.value $\alpha \in F$. Let $W = \{v \in V \mid y(v) = \alpha v\}$ be the α e-space of y. Then $\forall v \in W$, $\forall x \in L$, $\phi(x)(v) = x \cdot v \in W$ since

$$y\left(\phi(x)(v)\right) = \phi(x)\left(y(v)\right) + [y, \phi(x)](v)$$
$$= \phi(x)(\alpha v) = \alpha\, \phi(x)(v).$$

This makes W an L-submod. of <u>irred.</u> L-mod. V and $v_0 \in W \neq 0$ so $W = V$. $\square$

<u>Note</u>: For $ad : L \to gl(L)$, get L is an L-mod. where the L action on L is by $x \cdot y = [x, y] = ad_x(y)$. An L-submod. of L is an ideal of L, so L irred. as L-mod. means L is a simple Lie alg. and L s.s. $\Rightarrow$ L is completely reducible.

Constructing new L-modules from given ones.

For V an L-mod. the dual space $V^* = Hom(V, F)$ is an L-mod. under the L-action

$(x \cdot f)(v) = -f(x \cdot v)$, $\forall f \in V^*, \forall x \in L, \forall v \in V.$

Axioms M1, M2 are easily checked. For M3:

$$([x,y] \cdot f)(v) = -f([x,y] \cdot v) = -f(x \cdot (y \cdot v) - y \cdot (x \cdot v))$$

$$= -f(x \cdot (y \cdot v)) + f(y \cdot (x \cdot v))$$

$$= (x \cdot f)(y \cdot v) - (y \cdot f)(x \cdot v) = -(y \cdot (x \cdot f))(v) + (x \cdot (y \cdot f))(v)$$

$= (x \cdot (y \cdot f) - y \cdot (x \cdot f))v$, $\forall v \in V$ means

$[x,y] \cdot f = x \cdot (y \cdot f) - y \cdot (x \cdot f)$, $\forall f \in V^*$, is M3.

Let V, W be L-modules. The tensor
product $V \otimes W$ is the vector space spanned by "basic tensors" $\{v \otimes w \mid v \in V, w \in W\}$ where the tensor is bilinear and $\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w)$, $\forall \alpha \in F$. If V has basis $S = \{v_1, \dots, v_m\}$ and W has basis $T = \{w_1, \dots, w_n\}$ then $V \otimes W$ has basis $S \otimes T = \{v_i \otimes w_j \mid 1 \le i \le m, 1 \le j \le n\}$.

To make $V \otimes W$ an L-module we define

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w), \ \forall x \in L.$$

Check M3: $[x,y] \cdot (v \otimes w) = ([x,y] \cdot v) \otimes w + v \otimes ([x,y] \cdot w)$

$$= \big(x \cdot (y \cdot v) - y \cdot (x \cdot v)\big) \otimes w + v \otimes \big(x \cdot (y \cdot w) - y \cdot (x \cdot w)\big)$$

$$= \big(x \cdot (y \cdot v)\big) \otimes w - \big(y \cdot (x \cdot v)\big) \otimes w + v \otimes \big(x \cdot (y \cdot w)\big) - v \otimes \big(y \cdot (x \cdot w)\big)$$

Compare that to: $x\cdot(y\cdot(v\otimes w))-y\cdot(x\cdot(v\otimes w))$

$= x\cdot((y\cdot v)\otimes w+v\otimes(y\cdot w))-y\cdot((x\cdot v)\otimes w+v\otimes(x\cdot w))$

$=(x\cdot(y\cdot v))\otimes w+(y\cdot v)\otimes(x\cdot w)+(x\cdot v)\otimes(y\cdot w)+v\otimes(x\cdot(y\cdot w))$

$-(y\cdot(x\cdot v))\otimes w-(x\cdot v)\otimes(y\cdot w)-(y\cdot v)\otimes(x\cdot w)-v\otimes(y\cdot(x\cdot w))$

we see that 4 terms cancel (middle of each row) leaving the 4 terms that match the last line on the last page. M1 and M2 are clear.

Let V and W be L-modules. The vector space $Hom(V,W)=\{f:V\to W \mid f \text{ is linear}\}$ is an L-mod. if we define the action of $x\in L$ on $f\in Hom(V,W)$ by $(x\cdot f)(v)=x\cdot(f(v))-f(x\cdot v)$, $\forall v\in V$.

<u>Exercise</u>: Check that this action satisfies M3.