Note. Since $V^* = Hom(V, F)$, if we view F as a 1-dim'l trivial L-mod. then the term $x\cdot(f(v)) = 0$ in the above formula, reducing it to the formula on page 124 for the L-action on V^*.

There is an isom. of vector spaces $\Phi: V^* \otimes W \longrightarrow Hom(V, W)$ determined by $\Phi(f \otimes w) = f_w$ where $\forall v \in V$, $f_w(v) = f(v)w$.

Using bases $S = \{v_1, \dots, v_n\}$ of V, dual basis $S^* = \{f_1, \dots, f_n\}$ of V^*, so $f_i(v_j) = \delta_{ij}$, and basis $T = \{w_1, \dots, w_m\}$ of W. Show that Φ is <u>onto</u>. Since $\dim(V^* \otimes W) = nm = \dim(Hom(V, W))$, Φ is isom.

Th. The vector space isom. $\Phi : V^* \otimes W \to \mathrm{Hom}(V,W)$ is an L-module hom, so it is an L-mod. isom.

Pf. Using the formulas for the L action on the two spaces, we must show that $\forall x \in L$,

$$\Phi(x\cdot(f\otimes w)) = x\cdot\Phi(f\otimes w).$$

Apply both sides to any $v \in V$, we need to show the results are equal in W. First,

$$x\cdot(f\otimes w) = (x\cdot f)\otimes w + f\otimes(x\cdot w) \text{ so}$$

$$\begin{aligned}\Phi(x\cdot(f\otimes w))(v) &= (x\cdot f)(v)w + f(v)(x\cdot w)\\ &= -f(x\cdot v)w + f(v)(x\cdot w) \quad \text{while}\end{aligned}$$

$$\begin{aligned}(x\cdot\Phi(f\otimes w))(v) &= (x\cdot f_w)(v) = x\cdot(f(v)w) - f_w(x\cdot v)\\ &= f(v)(x\cdot w) - f(x\cdot v)w. \quad \square\end{aligned}$$

<u>Cor.</u> $\Phi : V^* \otimes V \to End(V)$ is an L-mod. isom. when V is an L-mod.

<u>Example.</u> On page 21 of these notes we gave the explicit action of $L = sl(2,F) = \langle e, f, h \rangle$ on $F^2 = V$ with std. basis $\left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. study $V \otimes V$:

$$e \cdot \left\{\begin{matrix} v_1 \otimes v_1 \\ v_1 \otimes v_2 \\ v_2 \otimes v_1 \\ v_2 \otimes v_2 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ v_1 \otimes v_1 \\ v_1 \otimes v_1 \\ v_1 \otimes v_2 + v_2 \otimes v_1 \end{matrix}\right\}$$

$$f \cdot \left\{\begin{matrix} v_1 \otimes v_1 \\ v_1 \otimes v_2 \\ v_2 \otimes v_1 \\ v_2 \otimes v_2 \end{matrix}\right\} = \left\{\begin{matrix} v_2 \otimes v_1 + v_1 \otimes v_2 \\ v_2 \otimes v_2 \\ v_2 \otimes v_2 \\ 0 \end{matrix}\right\}$$

$$h \cdot \left\{\begin{matrix} v_1 \otimes v_1 \\ v_1 \otimes v_2 \\ v_2 \otimes v_1 \\ v_2 \otimes v_2 \end{matrix}\right\} = \left\{\begin{matrix} 2\, v_1 \otimes v_1 \\ 0 \\ 0 \\ -2\, v_2 \otimes v_2 \end{matrix}\right\}$$

$V \otimes V = V_{sym} \oplus V_{anti}$ where

$$V_{sym} = \langle v_1 \otimes v_1,\ v_1 \otimes v_2 + v_2 \otimes v_1,\ v_2 \otimes v_2 \rangle$$

$$V_{anti} = \langle v_1 \otimes v_2 - v_2 \otimes v_1 \rangle$$

Note: $v_1 \otimes v_1 \xrightarrow{f} v_1 \otimes v_2 + v_2 \otimes v_1 \xrightarrow{f} 2(v_2 \otimes v_2)$

h acts by: $\quad 2 \qquad\qquad 0 \qquad\qquad -2$

so V_{sym} is an irred. 3-dim'l L-mod.

But V_{anti} is a trivial 1-dim'l L-mod.

In the notation from pages 24-25, have irred. $sl(2,F)$-modules V_m for each $0 \le m \in \mathbb{Z}$ with $\dim(V_m) = m+1$. $F^2 = V_1$ and we just got

$$V_1 \otimes V_1 = V_2 \oplus V_0$$ was completely reducible.

dim: $2 \times 2 = 3 + 1$

Exercises: ① Show $V_1^* \cong V_1$.

② Show $\Phi : V_1^* \otimes V_1 \xrightarrow{\cong} End(V_1)$ gives the decomp. of $gl(V_1) = sl(V_1) \oplus F I_2$

Casimir element of a rep'n:

Let L be s.s. and $\phi: L \to gl(V)$ a <u>faithful</u> (inj.) rep'n.

Define $\beta: L \times L \to F$ bilinear symmetric form by $\beta(x,y) = Tr(\phi(x) \circ \phi(y))$. Then β is assoc. and $Rad(\beta) = S \trianglelefteq L$. Since $\phi(S) \cong S$ is solv. by Cartan's Criterion, L s.s. $\Rightarrow S = 0$, so β is <u>nondeg.</u> <u>Note</u>: For $\phi = ad$, $\beta = K$.

Let L be s.s. and let $\beta: L \times L \to F$ be <u>any</u> nondeg. symm. assoc. bil. form on L. Let $B = \{x_1, \ldots, x_n\}$ be a basis of L and let $B^* = \{y_1, \ldots, y_n\}$ be the <u>dual</u> basis s.t. $\beta(x_i, y_j) = \delta_{ij}$.

$\forall X \in L$, write $[X, X_i] = \sum_{j=1}^{n} a_{ij} X_j$ and $[X, Y_k] = \sum_{j=1}^{n} b_{kj} Y_j$. We have

$$\beta([X, X_i], Y_k) = \sum_{j=1}^{n} a_{ij}\, \beta(X_j, Y_k) = \sum_{j=1}^{n} a_{ij} \delta_{jk} = a_{ik}$$

$$= \beta(-[X_i, X], Y_k) = -\beta(X_i, [X, Y_k]) = -\sum_{j=1}^{n} b_{kj}\, \beta(X_i, Y_j)$$

$$= -\sum_{j=1}^{n} b_{kj} \delta_{ij} = -b_{ki}$$ so $a_{ik} = -b_{ki}$ says the matrix ${}_B[ad_X]_B$ representing ad_X w.r.t. B is $A^T = [a_{ij}]^T$ and ${}_{B^*}[ad_X]_{B^*} = [b_{ij}]^T$ are related as neg. transposes of each other.

For $\phi : L \to \mathfrak{gl}(V)$ define

$$C_\phi(\beta) = \sum_{i=1}^{n} \phi(x_i) \circ \phi(y_i) \in End(V) \quad \text{where}$$

$B = \{x_1, \dots, x_n\}$ and $B^* = \{y_1, \dots, y_n\}$ are dual bases of L w.r.t. β. We have the identity in $End(V)$,

$$\begin{aligned} [x, yz] &= x(yz) - (yz)x = xyz - yxz + yxz - yzx \\ &= (xy - yx)z + y(xz - zx) \\ &= [x, y]z + y[x, z]. \end{aligned}$$

Along with $a_{ik} = -b_{ki}$ from above, we get

$$\begin{aligned} [\phi(x), C_\phi(\beta)] &= \sum_{i=1}^{n} [\phi(x), \phi(x_i)]\phi(y_i) + \sum_{i=1}^{n} \phi(x_i)[\phi(x), \phi(y_i)] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\phi(x_j)\phi(y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\phi(x_i)\phi(y_j) = 0 \end{aligned}$$

$\boxed{\text{So } C_\phi(\beta) \text{ commutes with } \phi(L)}$

Summarizing: For faithful $\phi : L \to gl(V)$
with nondeg. trace form $\beta(x,y) = Tr(\phi(x) \circ \phi(y))$
and some fixed basis $\{x_1, \dots, x_n\}$ call $C_\phi = C_\phi(\beta)$
the Casimir element of ϕ. $C_\phi \in End(V)$ s.t.
$[\phi(x), C_\phi] = 0$, $\forall x \in L$, so if V is an irred
L-mod, Schur's Lemma says $C_\phi = \alpha I_V$ for
some $\alpha \in F$. Then $Tr(C_\phi) = Tr(\alpha I_V) = \alpha \dim(V)$
but $Tr(C_\phi) = Tr\left(\sum_{i=1}^n \phi(x_i) \circ \phi(y_i)\right) = \sum_{i=1}^n \beta(x_i, y_i)$
$= \sum_{i=1}^n \delta_{ii} = n = \dim(L)$. So $\alpha = \dim(L)/\dim(V)$,
independent of choice of basis for L.