Ex. On page 110, we computed the trace form

$$\beta(x_i, x_j) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Tr(x_i x_j) \text{ for } L = sl(2, F)$$

using the natural rep'n on F^2 and basis $x_1 = e$, $x_2 = f$, $x_3 = h$. So the dual basis is $y_1 = f$, $y_2 = e$, $y_3 = \frac{1}{2}h$ and then

$$c_\phi = ef + fe + \frac{1}{2}h^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} = 3/2\, I_{F^2}$$

computed using 2×2 matrices representing these elements of $End(F^2)$. Of course, $\alpha = 3/2 = \dfrac{\dim(L)}{\dim(V)}$.

See Humphreys (p. 28) for remarks on what to do when ϕ is not faithful.

Weyl's Thm. on complete reducibility:

Lemma: Let $\phi: L \to gl(V)$ for L s.s. Then $\phi(L) \subseteq sl(V)$, and so $\dim(V) = 1 \Rightarrow \phi(L) = 0$.

Pf. For L s.s. we know $L = [L, L]$ so $\phi(L) = [\phi(L), \phi(L)]$ and all commutators have Trace 0 so $\phi(L) \subseteq sl(V)$.

Th (Weyl). Let $\phi: L \to gl(V)$ for L s.s., $\dim(V) < \infty$. Then ϕ is completely reducible, that is, L-mod. V is comp. redcible.

Pf. First consider case when $\exists\, W \leq V$ L-submod. such that $\dim(W) = \dim(V) - 1$, so $\dim(V/W) = 1$ and V/W is a trivial 1-dim'l L-mod. by lemma.

Write $V/W \cong F$, so get exact sequence

$$0 \to W \to V \to F \to 0.$$

we do a proof by induction on $\dim(V)$ of this special case. The inductive hypothesis is:
$\forall\, \phi : L^{s.s.} \to \mathfrak{gl}(\mathcal{U})$ s.t. $\dim(\mathcal{U}) < \dim(V)$ and s.t. $\exists\, \mathcal{U}' \leq \mathcal{U}$ L-submod with $\dim(\mathcal{U}/\mathcal{U}') = 1$, then $\mathcal{U}'$ has a complement in $\mathcal{U}$.

Let $0 \neq W' \lneq W$ L-submod. Then get induced exact seq. $0 \to W/W' \to V/W' \to F \to 0$ of L-modules.
By ind. with $\mathcal{U} = V/W'$, $\mathcal{U}' = W/W'$, $\exists$ complement (1-dim'l) $\tilde{W}/W' \leq V/W'$ s.t. $V/W' = W/W' \oplus \tilde{W}/W'$.

This means we get an exact seq. $0 \to W' \to \tilde{W} \to F \to 0$ with $\dim(W') < \dim(W)$, so ind. hypothesis gives a (1-dim'l) complement to W' in $\tilde{W}$, say $\tilde{W} = W' \oplus X$.
$W/W' \cap \tilde{W}/W' = 0/W'$ means $W \cap \tilde{W} \leq W'$ so $W \cap X = 0$.

$\dim(W \oplus X) = \dim(W) + \dim(X) = \dim(W) + 1 = \dim(V)$

so $W \oplus X = V$ gives an L-mod. compl. to W in V in the case when W is not irred.

We still need to cover the case when W is irred. We also assume ϕ is a faithful rep'n of L on V since otherwise we are looking at a faithful rep'n of a s.s. subalg. of L on V.

Let $c = c_\phi$ be the Casimir elt. so $[c, \phi(L)] = 0$. This means $c \circ \phi(x) = \phi(x) \circ c$ so c is an L-mod. map (hom.), $c : V \to V$ and $c(W) \subseteq W$ and $\mathrm{Ker}(c) \leq V$ is an L-submod. (W irred $\Rightarrow c|_W = \alpha I_W$)

But $\phi(L)(V) \subseteq W$ since L's action on V/W is triv. so c, as a sum of products $\phi(x_i)\phi(y_i)$, does the same, c acts on V/W by 0, has Trace 0 on V/W.

W irred $\Rightarrow c|_W = \alpha I_W$ and $\alpha = \frac{\dim(L)}{\dim(W)} \neq 0$.

This means $V = W \oplus \mathrm{Ker}(c)$ with $\dim(\mathrm{Ker}(c)) = 1$ and $\mathrm{Ker}(c)$ is an L-submod., the desired complement of W in V in this case.

<u>General Case</u>: Let $0 \neq W \subsetneqq V$ L-submod. so

$0 \to W \to V \to V/W \to 0$ is an exact seq. of L-mods.

$\mathrm{Hom}(V, W)$ is an L-mod. (Exercise, p. 126)

Let $\mathcal{V} = \{ f \in \mathrm{Hom}(V,W) \mid f|_W = \alpha I_W \text{ for some } \alpha \in F \}$.

Check that $\mathcal{V}$ is an L-submod: $\forall x \in L, w \in W$,

$(x \cdot f)(w) = x \cdot f(w) - f(x \cdot w) = x \cdot (\alpha w) - \alpha(x \cdot w) = 0$

so $(x \cdot f)|_W = 0$.

Let $\mathcal{W} = \{ f \in \mathcal{V} \mid f|_W = 0 \}$, so $\mathcal{W}$ is an L-submod.

and we just saw that $L(\mathcal{V}) \subseteq \mathcal{W}$.

Claim: $\dim(\mathcal{V}/\mathcal{W}) = 1$.

Pf. Let $\Psi : \mathcal{V} \to F$ be the map s.t. $\Psi(f) = \alpha$ if $f|_W = \alpha I_W$. Ψ is clearly linear with $\mathrm{Ker}(\Psi) = \mathcal{W}$, so $\dim(\mathcal{V}) = \dim(\mathcal{W}) + \dim(\mathrm{Range}(\Psi))$.

To get $\dim(\mathrm{Range}(\Psi)) = 1$ we only need to find some $f \in \mathcal{V}$ with $\Psi(f) = \alpha \neq 0$. That is easy to do since $W \leq V$ as a subspace, we can extend a basis $\{w_1, \dots, w_m\}$ for W to a basis $\{w_1, \dots, w_m, v_1, \dots, v_{n-m}\}$ of V and define $f : V \to W$ to be the lin. map s.t. $f(w_i) = \alpha w_i$, $f(v_j) = 0$.

So now we have an exact seq. of L-mods.

$$0 \to \mathcal{W} \to \mathcal{V} \to F \to 0$$

The first part of the proof tells us that $\mathcal{V}$ has a 1-dim'l L-submod. complement to $\mathcal{W}$. Let $f: V \to W$ be a basis for it chosen s.t. $f|_W = 1\, I_W$. The trivial action of L on this 1-dim'l L-mod. means $\forall v \in V$, $\forall x \in L$,

$0 = (x \cdot f)(v) = x \cdot (f(v)) - f(x \cdot v)$, so

$x \cdot f(v) = f(x \cdot v)$ which means f is an L-mod map, so $\mathrm{Ker}(f) \leq V$ is an L-submod. of V.

Finally, $V = W \oplus \mathrm{Ker}(f)$ achieves the goal. $\square$

Weyl's Thm. can be used to prove that the abstract Jordan decomposition $X = X_s + X_n$ is compatible with J-decompositions coming from linear rep'ns of L.