Preservation of Jordan decomposition:

Th. Let $L \subseteq gl(V)$ be a s.s. Lie algebra, $\dim(V)<\infty$. Then L contains the s.s. and nilp. parts in $gl(V)$ of all its elements. Therefore, the abstract and usual linear algebra Jordan decompositions in L coincide.

Pf. Uniqueness for both types of J-decomp. shows the last part of the Thm. follows from the first part.

Let $x \in L$ and let $x = x_s + x_n$ be the J-decomp. of x in $gl(V)$. Need to show $x_s, x_n \in L$. Since $ad_x(L) \subseteq L$ and we know ad_{x_s} and ad_{x_n} are polys. in ad_x (with 0 constant term in each poly.) we get $ad_{x_s}(L) \subseteq L$ and $ad_{x_n}(L) \subseteq L$ where we understand these maps as being in $gl(V)$, that is, $ad_{\circ}: V \to V$.

This means $x_s, x_n \in N_{gl(V)}(L) = N \leq gl(V)$
and N is a Lie subalg. of $gl(V)$ in which $L \trianglelefteq N$.
Note that $N \neq L$ since $L \subseteq sl(V)$ but the scalar operators $\alpha I_V \in N$ but $\alpha I_V \notin L$ for $Tr(\alpha I_V) = \alpha \dim(V) \neq 0$.

$\forall W \leq V$ L-submod. define
$L_W = \{y \in gl(V) \mid y(W) \subseteq W \text{ and } Tr(y|_W) = 0\}$.
Then $L_V = sl(V)$, for example. Since $L = [L, L]$ for L s.s., and W any L-mod. means $L(W) \subseteq W$, we see $L \leq L_W$ for any L-mod. W.

Let $L' = \left(\bigcap_{\substack{W\ L\text{-mod} \\ \text{in } V}} L_W\right) \cap N$ so $L \trianglelefteq L' \leq N$ but $\alpha I_V \notin L'$ for $\alpha \neq 0$. Also, $\forall x \in L$, $x_s, x_n \in L'$.

Claim: $L = L'$. Since L' is a fin. dim'l L-mod.
Weyl's Thm says $L' = L \oplus M$ for some complement.
L-mod. M. We know $[L, L'] \subseteq L$ from $L' \subseteq N$
so the L action on M is trivial, $[L, M] \subseteq L \cap M = 0$.
Let $W \subseteq V$ be any irred L-submod. $\forall y \in M$,
$[L, y] = 0$ so by Schur's lemma, $y|_W = \alpha I_W$ is scalar,
but $y \in L_W$ means $Tr(y|_W) = 0$ so $\alpha = 0$.
By Weyl's Thm, V is a direct sum of irred. L-mods
so $y = 0$ is the zero operator on all of V. Then
$M = 0$ and $L = L'$ finishes the proof. $\square$

Cor: Let L be s.s., $\phi: L \to gl(V)$ for $\dim(V) < \infty$. If $x = s + n$ is the abstract J-decomp. of $x \in L$, then $\phi(x) = \phi(s) + \phi(n)$ is the usual J-decomp. of $\phi(x)$ in $gl(V)$.

Pf. Lie alg. $\phi(L)$ is spanned by the e-vectors of $ad_{\phi(s)}: \phi(L) \to \phi(L)$ since L is spanned by the e-vectors of $ad_s: L \to L$, so $ad_{\phi(s)}$ is s.s. Similarly, $ad_{\phi(n)}: \phi(L) \to \phi(L)$ is nilp. since $ad_n: L \to L$ is nilp., and $[ad_{\phi(s)}, ad_{\phi(n)}] = ad_{\phi[s,n]} = 0$. So $\phi(x) = \phi(s) + \phi(n)$ is the abstract J-decomp. of $\phi(x)$ in s.s. Lie alg. $\phi(L)$. The Thm. then gives the Cor. $\square$