$L = sl(2,F)$ representations:

Assume V is a fin. dim'l L-module.
$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in L$ is s.s. so the last Cor. says h acts diagonally on V. (F alg. closed so all of its e-values are in F.) So get e-space decomp. $V = \bigoplus V_\lambda$ for
$V_\lambda = \{ v \in V \mid h \cdot v = \lambda v \} \neq 0$, called the λ __weight space__ of V and those $\lambda \in F$ are called the __weights__ of V.

__Lemma:__ If $v \in V_\lambda$ then $e \cdot v \in V_{\lambda+2}$, $f \cdot v \in V_{\lambda-2}$.

__Pf.__ $h \cdot (e \cdot v) = e \cdot (h \cdot v) + [h,e] \cdot v = e \cdot (\lambda v) + 2 e \cdot v$
$= (\lambda + 2)\, e \cdot v$ and similarly, since $[h,f] = -2f$,
$h \cdot (f \cdot v) = (\lambda - 2)\, f \cdot v$. $\square$

Note: $\dim(V) < \infty$ so $\{\lambda \in F \mid V_\lambda \neq 0\}$ is a fin. set, so the actions of e and f on V are by nilp. operators (already seen in last Cor.). $\exists \lambda \in F$ weight of V s.t. $\lambda + 2$ is <u>not</u> a wt. of V, that is, $e \cdot V_\lambda = 0$. Any $0 \neq v \in V_\lambda$ s.t. $e \cdot v = 0$ is called a maximal (highest) vector of wt. λ.

Classification of irred. L-modules:

Assume V is an irred. L-mod. Choose a maximal vector $v_0 \in V_\lambda$ and let $v_{-1} = 0$ and for $i \geq 0$ let $v_i = \frac{1}{i!} f^i \cdot v_0 = \frac{1}{i!} \underbrace{f \cdot (f \cdot (\cdots f}_{i\text{-times}} \cdot v_0))$. Then we get the following formulas for the actions of e, f, h on each v_i:

Lemma: (a) $h \cdot v_i = (\lambda - 2i)\, v_i$

(b) $f \cdot v_i = (i+1)\, v_{i+1}$

(c) $e \cdot v_i = (\lambda - i + 1)\, v_{i-1}$ for $i \geq 0$.

Pf. (a) For $i=0$, $h \cdot v_0 = \lambda v_0$ is given. For $i=1$, $h \cdot (f \cdot v_0) = (\lambda - 2)\, f \cdot v_0$ was done in previous Lemma. For $i>1$, use induction:

$$h \cdot (f^i \cdot v_0) = h \cdot (f \cdot (f^{i-1} \cdot v_0)) = f \cdot (h \cdot (f^{i-1} \cdot v_0)) +$$
$$[h,f] \cdot (f^{i-1} \cdot v_0) = f \cdot \big(\lambda - 2(i-1)\big) f^{i-1} \cdot v_0 - 2 f \cdot (f^{i-1} \cdot v_0)$$
$$= \big(\lambda - 2(i-1) - 2\big)(f^i \cdot v_0) = (\lambda - 2i)\, f^i \cdot v_0.$$

Mult. by $\frac{1}{i!}$ to get $h \cdot v_i = (\lambda - 2i)\, v_i$.

(b) $f \cdot v_i = f \cdot \frac{1}{i!}\left(f^i \cdot v_0\right) = \frac{1}{i!} f^{i+1} \cdot v_0$

$= (i+1) \frac{1}{(i+1)!} f^{i+1} \cdot v_0 = (i+1) v_{i+1}.$

(c) For $i = 0$, $e \cdot v_0 = (\lambda+1) v_{-1} = 0$ (by def. of v_{-1}) is true since v_0 is maximal. By induction,

$i\, e \cdot v_i = e \cdot (f \cdot v_{i-1}) = f \cdot (e \cdot v_{i-1}) + [e,f] \cdot v_{i-1}$

$= f \cdot (\lambda - (i-1) + 1) v_{i-2} + h \cdot v_{i-1}$

$= (\lambda - i + 2) f \cdot v_{i-2} + (\lambda - 2(i-1)) v_{i-1}$

$= (\lambda - i + 2)(i-1) v_{i-1} + (\lambda - 2i + 2) v_{i-1}$

$= \left(i(\lambda - i + 2) - (\lambda - i + 2) + (\lambda - 2i + 2)\right) v_{i-1}$

$= \left(i(\lambda - i + 2) - i\right) v_{i-1} = i(\lambda - i + 1) v_{i-1}$. Divide by i. $\square$

Note: $\{v_0, v_1, v_2, \dots \mid v_i \neq 0\}$ is **indep. set**
since they are nonzero vectors in distinct e-spaces. Let $0 \leq m \in \mathbb{Z}$ be the least s.t. $v_m \neq 0$ but $v_{m+1} = 0$ (so $v_{m+i} = 0, \forall i \geq 0$). The subspace of V spanned by $\{v_0, v_1, \dots, v_m\}$ is closed under the actions of e, f and h, so is an L-submod of irred. V, and is nontriv. since $v_0 \neq 0$, so $V = \langle v_0, \dots, v_m \rangle$, $\dim(V) = m+1$.
Lemma above gives the matrices representing the actions of e, f and h on V. For h it is diagonal, for e it is upper triangular, for f it is lower triangular (nilp. for e, f).

From (c) with $i = m+1$ get

$0 = e \cdot v_{m+1} = (\lambda - m) v_m$ with $v_m \neq 0$ so $\underline{\lambda = m}$

that is, the weight λ of a max. vector is the non-negative integer $m = \dim(V) - 1$.

Also, $\dim(V_{m-2i}) = 1$ for each $0 \leq i \leq m$.

<u>Th</u> (Summary) Let V be an irred L-mod for $L = sl(2, F)$. (a) $V = \bigoplus_{i=0}^{m} V_{m-2i}$ e-spaces for the action of h with e.values $m, m-2, \ldots, -(m-2), -m$.

(b) V contains a maximal vector $v_0 \neq 0$ of weight m, $V_m = F v_0$.

(c) The action of L on V is given by the formulas in the Lemma w.r.t. basis $\{v_i \mid 0 \leq i \leq m\}$, so V is uniquely det'd by m.

<u>Cor</u>. Let V be any fin. dim'l $\mathfrak{sl}(2,F)$-mod. The e.values of h on V are all in $\mathbb{Z}$, for each e.value μ there is an e.value $-\mu$, $\dim(V_\mu) = \dim(V_{-\mu})$, and if V is decomposed as a direct sum of irred. L-mods, the number of summands is $\dim(V_0) + \dim(V_1)$.

<u>Pf</u>. Clear if $V = 0$. For $\dim(V) \geq 1$, use Weyl's Th. to decompose V into a direct sum of irred. L-mods, each of which has the weight space structure given in Thm above. For each irred. summand either all weights are even or all are odd, contributing one to either V_0 or V_1, respectively. $\square$

<u>Notation:</u> For $0 \leq m \in \mathbb{Z}$, let $V(m)$ be the irred. $sl(2,F)$-mod. with basis $\{v_0, \dots, v_m\}$ and actions of e, f, h given in the Lemma. By uniqueness, this must be isom. to the $sl(2,F)$ irred module given on page 25.

<u>Discussion of $\mathbb{Z}_2$ symmetry of $V(m)$:</u>

Let $\phi : L \to gl(V(m))$ be the irred. rep'n of $L = sl(2,F)$ on the <u>irrep.</u> $V(m)$ with highest wt. m. $\phi(e)$ and $\phi(f)$ are nilp. so have $\tau = \exp(\phi(e)) \exp(\phi(-f)) \exp(\phi(e)) \in Aut(V(m))$ (recall discussion on p. 46). Let $m > 0$ so ϕ faithful.

As discussed on pages 49-50, we have

$$\tau\,\phi(h)\,\tau^{-1} = \exp(\mathrm{ad}_{\phi(e)})\exp(\mathrm{ad}_{\phi(-f)})\exp(\mathrm{ad}_{\phi(e)})\,\phi(h)$$

and $\phi(L)\cong L$. Since $\sigma = \exp(\mathrm{ad}_e)\exp(\mathrm{ad}_{-f})\exp(\mathrm{ad}_e) \in \mathrm{Int}(L)$ acts on L by $\sigma(e) = -f$, $\sigma(f) = -e$ and $\sigma(h) = -h$, we get $\phi(\sigma(h)) = -\phi(h) = \tau\,\phi(h)\,\tau^{-1}$, that is, $\tau\,\phi(h) = -\phi(h)\,\tau$. This gives

$$\phi(h)\,\tau(v_i) = -\tau(\phi(h)\,v_i) = -\tau(m-2i)v_i \quad \text{so}$$

$\tau(v_i)$ has weight $-(m-2i)$ so $\tau(v_i) \in F v_{m-i}$.

If we graph the weights on the real line,

then τ is an "involution" exchanging weight spaces $V_\mu \xleftrightarrow{\tau} V_{-\mu}$.