Root space decomposition:

Example: $L = \mathfrak{sl}(3,F)$. Let $h_1 = E_{11} - E_{22}$, $h_2 = E_{22} - E_{33}$ so $[h_1, h_2] = 0$ and let $H = \langle h_1, h_2 \rangle \leq L$, an abelian subalgebra of s.s. elements whose ad action on L is simultaneously diag-able. Using the bracket formula from page 6 of these notes,

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

and the notations from page 7:

$e_1 = E_{12}$, $e_2 = E_{23}$, $f_1 = E_{21}$, $f_2 = E_{32}$

we compute $[h_i, e_j]$ and $[h_i, f_j]$.

$$[h_1, e_1] = [E_{11} - E_{22}, E_{12}] = [E_{11}, E_{12}] - [E_{22}, E_{12}] = E_{12} - (-E_{12}) = 2E_{12} = 2e_1$$

$$[h_1, e_2] = [E_{11} - E_{22}, E_{23}] = [E_{11}, E_{23}] - [E_{22}, E_{23}] = 0 - E_{23} = -e_2$$

$$[h_2, e_1] = [E_{22} - E_{33}, E_{12}] = -E_{12} = -e_1$$

$$[h_2, e_2] = [E_{22} - E_{33}, E_{23}] = E_{23} - (-E_{23}) = 2E_{23} = 2e_2$$

So $\forall h = a_1 h_1 + a_2 h_2 \in H$, $[h, e_1] = [a_1 h_1 + a_2 h_2, e_1] = (2a_1 - a_2) e_1$ and $[h, e_2] = (-a_1 + 2a_2) e_2$.

Define $\alpha_1, \alpha_2 \in H^* = Lin(H, F)$ by $\alpha_1(a_1 h_1 + a_2 h_2) = 2a_1 - a_2$ and $\alpha_2(a_1 h_1 + a_2 h_2) = -a_1 + 2a_2$ so $[h, e_j] = \alpha_j(h) e_j$ for $j = 1, 2$. Also, these "roots" $\alpha_1, \alpha_2 \in H^*$ are defined by their values on $h_1, h_2 \in H$, recorded in the matrix $[\alpha_i(h_j)] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Let $e_3 = [e_1, e_2] = [E_{12}, E_{23}] = E_{13}$ and compute $[h, e_3] = [h, [e_1, e_2]] = [[h, e_1], e_2] + [e_1, [h, e_2]] = \alpha_1(h)[e_1, e_2] + \alpha_2(h)[e_1, e_2] = (\alpha_1 + \alpha_2)(h) e_3$ where $(\alpha_1 + \alpha_2)(h) = (2a_1 - a_2 - a_1 + 2a_2) = a_1 + a_2$.

Similarly, $[h, f_j] = -\alpha_j(h) f_j$ for $j = 1, 2$.

<u>Note</u>. $f_3 = [f_1, f_2] = [E_{21}, E_{32}] = -E_{31}$ and $[h, f_3] = -(\alpha_1 + \alpha_2)(h) f_3$

Let $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\} \subseteq H^*$ and for $\alpha \in \Phi$ let $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x,\ \forall h \in H\}$. Then

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$$

is called the root space (or Cartan) decomposition of L.

This generalizes to all $L = \mathfrak{sl}(n+1, F)$ where
$h_i = E_{ii} - E_{(i+1)(i+1)}$, for $1 \le i \le n$, spans H, and
$e_i = E_{i(i+1)}$, $f_i = E_{(i+1)i}$ and $\alpha_i \in H^*$ are determined
by $[h, e_j] = \alpha_j(h)e_j$ which are determined by the
matrix $[\alpha_i(h_j)]$. For $n > 2$, the "root system" Φ
is more complicated than the one above for $n = 2$,
containing an α for each E_{ij}, $1 \le i \ne j \le n+1$.

We will see how this structure arises in any s.s. Lie algebra.

Let L be a non-zero s.s. Lie algebra.
L must contain non-zero elements x s.t. the s.s. part
$x_s \neq 0$, otherwise Engel's Thm says L is nilp.
So L contains non-zero subalgebras of s.s. elements.

Def. A subalg. of L consisting of s.s. elts is called **toral**.

Lemma. Any toral subalg. of L is abelian.

Pf. Let $T \leq L$ be toral. Show that $\forall x \in T$,
$ad_x : T \to T$ is the zero map. ad_x is diag-able on T
so we need to show all its e-values are 0. Suppose
$\exists y \in T$, $y \neq 0$ s.t. $[x,y] = ay$ for $0 \neq a \in F$. Then
$ad_y : T \to T$, $ad_y(x) = [y,x] = -ay$ is an e-vector of ad_y
with e-value 0. But y is s.s. in T so T has a basis
of e.vectors for ad_y and x is a lin. comb. of such e.vectors

Then $ad_y(x)$ is a lin. comb. of those ad_y e-vectors which had non-zero e.values, contradicting $ad_y(x) = -ay$ an e-vector for ad_y with 0 e.value. $\square$

Def. Choose $H \leq L$ a *maximal* toral subalg. of L. By lemma, H is abelian, $ad_h : L \to L$ is a commuting set of s.s. endom's, $h \in H$, so they are simultaneously diag-able. Let $L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x, \ \forall h \in H\}$ for $\alpha \in H^*$ be the simult. e-space where the e.values of ad_h are given by lin. functional $\alpha(h)$. $L = \bigoplus_{\alpha \in H^*} L_\alpha$ but most $L_\alpha = \{0\}$. $L_0 = C_L(H) \geq H$.

$\{\alpha \in H^* \mid L_\alpha \neq 0, \ \alpha \neq 0\} = \Phi$ is called the *root system* of L w.r.t. H. It is a finite set since $\dim(L) < \infty$ and $L = C_L(H) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ is called a root space (Cartan) decomposition.

<u>Prop.</u> $\forall \alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$. If $x \in L_\alpha$ for $\alpha \neq 0$ then ad_x is nilp. If $\alpha, \beta \in H^*$ and $\alpha+\beta \neq 0$ then $K_L(L_\alpha, L_\beta) = 0$.

<u>Pf.</u> $\forall x \in L_\alpha$, $y \in L_\beta$, $h \in H$, we have $[h,[x,y]] = [[h,x],y] + [x,[h,y]] = \alpha(h)[x,y] + \beta(h)[x,y] = (\alpha+\beta)(h)[x,y]$ so $[x,y] \in L_{\alpha+\beta}$. Since Φ is finite, and $(ad_x)^n(y) \in L_{n\alpha+\beta}$ from first part, and $\alpha \neq 0$ so $\{n\alpha+\beta \mid n \in \mathbb{Z}\}$ is an infinite set, we must have $(ad_x)^n(y) = 0$ for some n. Applying this to each L_β in the Cartan decomp. of L (including $\beta = 0$), get ad_x is nilp.

Suppose $\alpha+\beta \neq 0$, so $\exists h \in H$ s.t. $(\alpha+\beta)(h) \neq 0$. For any $x \in L_\alpha$, $y \in L_\beta$, $K([h,x],y) = \alpha(h) K(x,y) = -K([x,h],y) = -K(x,[h,y]) = -\beta(h) K(x,y)$ so $(\alpha+\beta)(h) K(x,y) = 0$ so $K(x,y) = 0$. □

<u>Cor</u>. Restriction $K_{L_0}: L_0 \times L_0 \to F$ is non-degen.

<u>Pf</u>. For L s.s. we know $K: L \times L \to F$ is non-deg.
But from last Prop., $K(L_0, L_\alpha) = 0$ for all $\alpha \neq 0$.
If $z \in L_0$ and $K(z, L_0) = 0$ then $K(z, L) = 0$ so $z = 0$.
This says $\mathrm{Rad}(K|_{L_0}) = 0$ so the restriction is non-deg. ∎

<u>Lemma</u>. If $x, y \in \mathrm{End}(V)$, $\dim(V) < \infty$ and $xy = yx$ and y is nilp, then xy is nilp. so $\mathrm{Tr}(xy) = 0$.

<u>Pf</u>. y nilp. so $y^n = 0$ for some n so $(xy)^n = x^n y^n = 0$. □

<u>Prop</u>. Let H be a max. toral subalg of L. Then $H = C_L(H)$.

<u>Pf</u>. Let $C = C_L(H) = \{x \in L \mid [x, h] = 0, \forall h \in H\}$.

Step ①: <u>C contains the s.s. and nilp. parts of its elements.</u> $x \in C$ means $ad_x(H) = 0$. By Jordan Thm, part (c),