$(ad_x)_s(H) = 0 = (ad_x)_n(H)$ but $(ad_x)_s = ad_{x_s}$ and $(ad_x)_n = ad_{x_n}$ by results on abstract Jordan decomp, so $x_s, x_n \in C$.

Step ②: <u>All s.s. elts of C are in H.</u> If x is s.s. in C then $H + Fx$ is an abel, toral subalg of L (sum of commuting s.s. elts is s.s.). But H was <u>max.</u> toral subalg. so $H + Fx = H$ so $x \in H$.

Step ③: <u>$K|_H : H \times H \to F$ is nondeg.</u> Suppose $K(h, H) = 0$ for some $h \in H$. Show $h = 0$. If $x \in C$ is nilp then $[x, H] = 0$ and ad_x nilp. imply (by Lemma) $Tr(ad_x \circ ad_y) = 0$, $\forall y \in H$, so $K(x, H) = 0$. Now, $\forall x \in C$, by Step ①, $x_s, x_n \in C$ and by step ② $x_s \in H$, so $K(h, x) = K(h, x_s) + K(h, x_n) = 0$ so $K(h, C) = 0$ so $h = 0$ by Cor. above.

Step ④: C is nilp. If $x \in C$ is s.s. then $x \in H$ by Step ② so $ad_x : C \to C$ is the zero map is nilp.
If $x \in C$ is nilp. then $ad_x : C \to C$ is nilp.
For any $x \in C$ write $x = x_s + x_n$ where $x_s, x_n \in C$ by Step ①.
$ad_x = ad_{x_s} + ad_{x_n} : C \to C$ is sum of two commuting nilp. operators so ad_x is nilp. By Engel's Th, C is nilp.
Step ⑤: $H \cap [C,C] = 0$. $K(H, [C,C]) = K([H,C], C) = K(0, C) = 0$ so $\forall x \in H \cap [C,C]$, $K(H, x) = 0$ gives $x = 0$ by Step ③.
Step ⑥: C is abelian: If not, $[C,C] \neq 0$, so C nilp. from Step ④ gives $Z(C) \cap [C,C] \neq 0$ (Lemma on p.13 of Humphreys). Let $0 \neq z \in Z(C) \cap [C,C]$. If z were s.s. then by Step ②, $z \in H$, contradicting Step ⑤.
So $z = s + n$ with $n \neq 0$ (nilp. part) and $n \in C$ by Step ①.

By Jordan Decomp, ad_n is a poly in ad_z, but ad_z acts as 0 on C since $z \in Z(C)$ so ad_n also acts as 0 on C, meaning $n \in Z(C)$. Apply the lemma to get $K(n, C) = 0$, a contradiction to $K|_{C \times C}$ non-degen.

Step ⑦: <u>$C = H$</u>. If not $\exists\, 0 \neq x \in C$, x nilp. from Steps ① and ②. By Lemma and Step ⑥ we have $\forall y \in C$, $K(x, y) = Tr(ad_x \circ ad_y) = 0$ contradicts that $K|_{C \times C}$ is non-deg. $\blacksquare$

<u>Cor</u>: $K|_{H \times H} : H \times H \to F$ is non-deg.

<u>Note</u>: $K : H \times H \to F$ non-deg. means we have an <u>isom</u> between H and H^*. $\forall\, \phi \in H^*$, $\exists!\, t_\phi \in H$ s.t. $\phi(h) = K(t_\phi, h)$, $\forall h \in H$. $\forall\, t \in H$, define $\phi_t \in H^*$ by $\phi_t(h) = K(t, h)$, $\forall h \in H$.

Orthogonality Properties:

Prop. (a) Φ spans H^* (b) If $\alpha \in \Phi$ then $-\alpha \in \Phi$

(c) Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$. Then $[x,y] = K(x,y)t_\alpha$

(d) If $\alpha \in \Phi$ then $\dim([L_\alpha, L_{-\alpha}]) = 1$ with basis t_α.

(e) $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0 \quad \forall \alpha \in \Phi$.

(f) $\forall \alpha \in \Phi, \forall 0 \neq x_\alpha \in L_\alpha, \exists\, y_\alpha \in L_{-\alpha}$ s.t. $\{x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]\}$ is a basis of a simple subalg. of L isom. to $sl(2,F)$ by $x_\alpha \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y_\alpha \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h_\alpha \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(g) $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ and $h_{-\alpha} = -h_\alpha$.

Pf. (a) If $\langle \Phi \rangle \neq H^*$ then $\exists\, 0 \neq h \in H$ s.t. $\alpha(h) = 0$, $\forall \alpha \in \Phi$, so $[h, L_\alpha] = 0$, $\forall \alpha \in \Phi$. $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ and $[h,H]=0$ so $[h,L]=0$ says $h \in Z(L)$ contradicts L s.s.

(b) Let $\alpha \in \Phi$ but $-\alpha \notin \Phi$ so $L_{-\alpha} = 0$. Then $K(L_\alpha, L_\beta) = 0$ $\forall \beta \in H^*$ so $K(L_\alpha, L) = 0$ contradicts K non-degen.

(c) Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$, $h \in H$. Then
$K(h, [x,y]) = K([h,x], y) = \alpha(h) K(x,y) = K(t_\alpha, h) K(x,y)$
$= K(K(x,y) t_\alpha, h) = K(h, K(x,y) t_\alpha)$. So $[x,y] - K(x,y) t_\alpha \in H$
is orthog to all of H. $K|_{H \times H}$ non-deg. implies it is 0.

(d) From (c), $[L_\alpha, L_{-\alpha}] = \langle t_\alpha \rangle$ if it is $\neq 0$. For any
$0 \neq x \in L_\alpha$ if $K(x, L_{-\alpha}) = 0$ then $K(x, L) = 0$ contradicts
K non-deg., so $\exists\, 0 \neq y \in L_{-\alpha}$ s.t. $K(x,y) \neq 0$. From (c) get
$[x,y] = K(x,y) t_\alpha \neq 0$, so $\dim([L_\alpha, L_{-\alpha}]) = 1$.

(e) Suppose $\alpha(t_\alpha) = 0$ so $[t_\alpha, x] = 0 = [t_\alpha, y]$ $\forall x \in L_\alpha$,
$\forall y \in L_{-\alpha}$. Can find some such x and y with $K(x,y) = 1$
so $[x,y] = t_\alpha$. The subspace $S = \langle x, y, t_\alpha \rangle$ is a 3-dim'l
solvable Lie subalg. of L.

Under ad $: L \longrightarrow gl(L)$ (injective since L is s.s.) the image ad$(S) \cong S$ and $\forall \alpha \in [S,S] = \langle t_\alpha \rangle$ is nilp. by Cor. to Lie's Thm. (Humphreys, p.16). But $t_\alpha \in H$ means ad_{t_α} is s.s. as well as nilp, so $ad_{t_\alpha} = 0$ $[t_\alpha, L] = 0$, $t_\alpha \in Z(L) = 0$, contradiction.

(f) For $\alpha \in \Phi$ given $0 \neq X_\alpha \in L_\alpha$, $\exists\, Y_\alpha \in L_{-\alpha}$ s.t.

$$K(X_\alpha, Y_\alpha) = \frac{2}{K(t_\alpha, t_\alpha)}$$

since (e) holds, and $K(X_\alpha, L_{-\alpha}) \neq 0$

Let $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ so from (c), $[X_\alpha, Y_\alpha] = h_\alpha$. Also,

$$[h_\alpha, X_\alpha] = \alpha(h_\alpha) X_\alpha = \frac{2\alpha(t_\alpha)}{K(t_\alpha, t_\alpha)} X_\alpha = 2X_\alpha \text{ and}$$

$[h_\alpha, Y_\alpha] = -2Y_\alpha$. So $\langle X_\alpha, Y_\alpha, h_\alpha \rangle \cong sl(2,F)$

(g) h_α was defined in (f) so $h_{-\alpha} = 2t_{-\alpha}/K(t_{-\alpha}, t_{-\alpha})$ but $K(t_{-\alpha}, h) = -\alpha(h) = K(-t_\alpha, h)$ says $t_{-\alpha} = -t_\alpha$. $\square$