Integrality Properties:

For each pair of roots $\alpha, -\alpha \in \Phi$ let $S_\alpha \cong sl(2,F)$ be the subalg. of L with basis $\{x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]\}$ from last Prop. (f). Study L as an S_α-module using information about irred. S_α-modules and the complete reducibility of L as a direct sum of them.

Fix $\alpha \in \Phi$, let $M = H \bigoplus_{c \in F^*} L_{c\alpha}$. This subspace of L is an S_α-submod. The e-values (weights) of ad_{h_α} on M are 0 and $c\alpha(h_\alpha) = 2c$ if $L_{c\alpha} \neq 0$, and these must all be integers. So $c \in \frac{1}{2}\mathbb{Z}$. $H = Ker(\alpha) \oplus Fh_\alpha$ and $S_\alpha \cdot Ker(\alpha) = 0$. S_α is itself an irred S_α-submod of M, $S_\alpha = Fh_\alpha \oplus L_\alpha \oplus L_{-\alpha}$. So the 0 weight space of M, $M_0 = H$ is contained in $Ker(\alpha) \oplus S_\alpha$ a direct sum of irred. S_α-submodules. So no other irred. S_α-submods.

with weight 0 can occur in L, and the only even weights that occur are $0, 2, -2$.

Thus, if $\alpha \in \Phi$ then $2\alpha \notin \Phi$. Also $\frac{1}{2}\alpha \notin \Phi$ since otherwise $\alpha = 2(\frac{1}{2}\alpha) \notin \Phi$, so 1 is not a weight (e.value) of h_α in M. But any irred. S_α-submods of M with odd weights would include a wt. 1 e-space, so $M = H + S_\alpha$ ($H \cap S_\alpha = Fh_\alpha$)

$$= H \oplus L_\alpha \oplus L_{-\alpha} \text{ and } \dim(L_\alpha) = 1.$$

Let $\beta \neq \pm\alpha \in \Phi$, $K = \bigoplus_{i \in \mathbb{Z}} L_{\beta + i\alpha}$. $\beta + i\alpha \neq 0$, $\dim(L_{\beta+i\alpha}) = 1$ if $\beta + i\alpha \in \Phi$, 0 if not. K is an S_α-submod. whose weights are integers $\beta(h_\alpha) + 2i$ if $\beta + i\alpha \in \Phi$. $\beta(h_\alpha) \in \mathbb{Z}$ is either even or odd, so all $\beta(h_\alpha) + 2i$ have same parity as $\beta(h_\alpha)$.

K cannot be the (direct) sum of two or more irred. S_α-submods since $\dim(L_{\beta+i\alpha}) \leq 1$ and only one parity of weights occurs in K.
So K is an irred. S_α-mod. with highest wt. $\beta(h_\alpha)+2q$ and lowest wt. $\beta(h_\alpha)-2r$ if q is largest integer s.t. $\beta+q\alpha \in \Phi$ and r is largest integer s.t. $\beta - r\alpha \in \Phi$. $(q, r \geq 0)$
The wts of K form an arithmetic progression in steps of 2 from $m = \beta(h_\alpha)+2q$ to $-m = \beta(h_\alpha)-2r$.
So these form the <u>α-string through β</u>,
$\beta - r\alpha, \dots, \beta, \dots, \beta + q\alpha \in \Phi$ and $\beta(h_\alpha) = r - q$.
If $\alpha, \beta, \alpha+\beta \in \Phi$ then $ad_{L_\alpha}(L_\beta) = L_{\alpha+\beta} = [L_\alpha, L_\beta]$
Summarize these results in following Prop.

<u>Prop.</u> (a) $\alpha \in \Phi \Rightarrow \dim(L_\alpha) = 1$, $S_\alpha = L_\alpha \oplus L_{-\alpha} \oplus H_\alpha$ [17]
for $H_\alpha = [L_\alpha, L_{-\alpha}] = F h_\alpha$. $\forall\, 0 \neq x_\alpha \in L_\alpha$, $\exists!\, y_\alpha \in L_{-\alpha}$
s.t. $[x_\alpha, y_\alpha] = h_\alpha$.

(b) If $\alpha \in \Phi$ and $c\alpha \in \Phi$ for $c \in F$ then $c = \pm 1$.

(c) If $\alpha, \beta \in \Phi$ then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$
(Note: Call $\beta(h_\alpha)$ Cartan integers.)

(d) If $\alpha, \beta, \alpha+\beta \in \Phi$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$

(e) Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Let r, q be the largest integers (resp.) s.t. $\beta - r\alpha, \beta + q\alpha \in \Phi$. Then all $\beta + i\alpha \in \Phi$ for $-r \leq i \leq q$ and $\beta(h_\alpha) = r - q$.

(f) L is generated as a Lie alg. by root spaces L_α, $\alpha \in \Phi$.

Rationality Properties:

We have L s.s. Lie alg. over field F (alg. closed char. 0), H a max. toral subalg., $\Phi \subset H^*$ the roots of L w.r.t. H, $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ root space decomp.

$K|_{H \times H}$ nondeg. gives bijection (isom.) $\lambda \leftrightarrow t_\lambda$ $\forall \lambda \in H^*$, so define bil. form on H^* by

$(\lambda, \mu) = K(t_\lambda, t_\mu)$ $\forall \lambda, \mu \in H^*$.

Since $\langle \Phi \rangle = H^*$ can choose a basis of H^* to be some roots $T = \{\alpha_1, \dots, \alpha_\ell\}$, $\ell = \dim(H) =$ "$\mathrm{rank}(L)$". $\forall \beta \in \Phi$ write $\beta = \sum_{j=1}^{\ell} c_j \alpha_j$ for $c_j \in F$.

__Claim:__ $c_j \in \mathbb{Q}$ are rational numbers.

Pf. For $1 \le i \le \ell$ we have $(\beta, \alpha_i) = \sum_{j=1}^{\ell} c_j (\alpha_i, \alpha_j)$

so $\dfrac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)} = \sum_{j=1}^{\ell} \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} c_j$ is a linear system of ℓ equations in the ℓ variables $c_1, \dots, c_\ell$ and whose coeffs. are Cartan integers:

$\alpha_j(h_{\alpha_i}) = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ since $h_{\alpha_i} = \dfrac{2 t_{\alpha_i}}{(\alpha_i, \alpha_i)}$ and

$$\beta(h_{\alpha_i}) = \frac{2\beta(t_{\alpha_i})}{(\alpha_i, \alpha_i)} = \frac{2K(t_\beta, t_{\alpha_i})}{K(t_{\alpha_i}, t_{\alpha_i})} = \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}.$$

The matrix $\mathcal{A} = [(\alpha_i, \alpha_j)]$ is invertible since the form $(\cdot, \cdot)$ is non-deg. on H^* and $T = \{\alpha_1, \dots, \alpha_\ell\}$ is a basis of H^*. $\forall \lambda, \mu \in H^*$, $(\lambda, \mu) = [\lambda]_T^{tr} \mathcal{A} [\mu]_T$.

The integral matrix $A = [2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)]$ $\angle 174$
$= D\mathcal{A}$ for $D = \mathrm{diag}(2/(\alpha_1,\alpha_1), \dots, 2/(\alpha_\ell,\alpha_\ell))$
is also invertible and is the coeff. matrix of that lin. sys. $AC = B$ for $C = [\beta]_T = [c_1, \dots, c_\ell]^{tr}$
and $B = [2(\beta,\alpha_i)/(\alpha_i,\alpha_i)] \in \mathbb{Z}^\ell$.

So the solution $C = A^{-1}B$ has entries all rationals with denominators $\det(A)$ (Cramer's Rule).

Let $E_{\mathbb{Q}} = \mathbb{Q}\text{-span}\{\alpha_1, \dots, \alpha_\ell\} \subseteq H^*$ so $\dim_{\mathbb{Q}}(E_{\mathbb{Q}}) = \ell$
$= \dim_F(H^*)$. Humphreys claims to prove that
$(\cdot,\cdot): E_{\mathbb{Q}} \times E_{\mathbb{Q}} \to \mathbb{Q}$ is rational valued and positive definite. $\forall \lambda, \mu \in H^*$, $(\lambda,\mu) = K(t_\lambda, t_\mu) =$
$\mathrm{Tr}_L(\mathrm{ad}_{t_\lambda} \circ \mathrm{ad}_{t_\mu})$ and $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$.