ad_{t_λ} and ad_{t_μ} act diagonally by 0 on H and by $\alpha(t_\lambda)$ and $\alpha(t_\mu)$ on each L_α, $\alpha \in \Phi$, where $\dim(L_\alpha) = 1$, so $Tr_L(ad_{t_\lambda} \circ ad_{t_\mu}) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha \in \Phi} (\alpha,\lambda)(\alpha,\mu)$. This gives

$\forall \beta \in \Phi$, $(\beta,\beta) = \sum_{\alpha \in \Phi} (\alpha,\beta)^2$ so $\frac{1}{(\beta,\beta)} = \sum_{\alpha \in \Phi} \frac{(\alpha,\beta)^2}{(\beta,\beta)^2} \in \mathbb{Q}$ since $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ $\forall \alpha, \beta \in \Phi$, so $(\beta,\beta) \in \mathbb{Q}$.

Then $(\alpha,\beta) \in \frac{(\beta,\beta)}{2}\mathbb{Z} \subseteq \mathbb{Q}$. In particular, $(\alpha_i,\alpha_j) \in \mathbb{Q}$ shows that $(\cdot,\cdot)$ on $E_\mathbb{Q}$ is rational valued, non-deg. sym. bil. form. For $\lambda \in E_\mathbb{Q}$, $(\lambda,\lambda) = \sum (\alpha,\lambda)^2 \geq 0$ is sum of rationals squared, and $(\lambda,\lambda)=0$ implies $(\alpha,\lambda)=0$ for all $\alpha\in\Phi$ so $\lambda=0$. This means the form is positive def. on $E_{\mathbb{Q}}$.

Let $E_{\mathbb{R}}=\mathbb{R}\text{-span}\{\alpha_1,\dots,\alpha_\ell\}=\mathbb{R}\otimes_{\mathbb{Q}}E_{\mathbb{Q}}$ the real vector space, with the sym. pos. def. bil. form (α_i,α_j) as above. $E=E_{\mathbb{R}}$ is a Euclidean space, $\dim_{\mathbb{R}}(E)=\ell$ in which we have the following.

<u>Th</u>: With L, H, Φ, E as defined above, we have

(a) $\langle\Phi\rangle=E$, $0\notin\Phi$

(b) If $\alpha\in\Phi$ then $-\alpha\in\Phi$, but $r\alpha\in\Phi\Rightarrow r=\pm1$

(c) If $\alpha,\beta\in\Phi$ then $\beta-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha\in\Phi$

(d) If $\alpha,\beta\in\Phi$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}$.

Root Systems:

Let E be a Euclidean space, that is, a real vectorspace of finite dimension with a pos. def. sym. bil. form (α,β).

Def: A reflection in E is a lin. map $\sigma_\alpha : E \to E$ which fixes pointwise a hyperplane, $P_\alpha = \{\beta \in E \mid (\beta,\alpha)=0\}$ and s.t. $\sigma_\alpha(\alpha) = -\alpha$, so $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$.

Fact: Any reflection σ_α is an invertible orthogonal trans. on E, that is, $(\sigma_\alpha(x), \sigma_\alpha(y)) = (x,y)$ $\forall x,y \in E$. Also, $\sigma_\alpha^{-1} = \sigma_\alpha$.

Notation: Let $\langle \beta,\alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ $\forall \alpha,\beta \in E$, $\alpha \neq 0$.

Lemma: Let Φ be finte set that spans E. Suppose $\forall \alpha \in \Phi$, $\sigma_\alpha(\Phi) \subseteq \Phi$. If $\sigma \in GL(E)$ leaves Φ invar. and fixes pointwise a hyperplane P and $\sigma(\alpha) = -\alpha$ for some $0 \neq \alpha \in \Phi$, then $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

<u>Pf</u>. Let $\tau = \sigma\sigma_\alpha$ so $\tau(\Phi) = \Phi$, $\tau(\alpha) = \alpha$ so $\tau|_{\mathbb{R}\alpha} = I_{\mathbb{R}\alpha}$. The quotient space $E/\mathbb{R}\alpha$ can be viewed as either $\{v+\mathbb{R}\alpha \mid v \in P\}$ or $\{v+\mathbb{R}\alpha \mid v \in P_\alpha\}$ since $E = P \oplus \mathbb{R}\alpha = P_\alpha \oplus \mathbb{R}\alpha$. From the first point of view, σ acts as the identity on $E/\mathbb{R}\alpha$, from the second point of view, σ_α acts as the id. on $E/\mathbb{R}\alpha$ so τ acts on $E/\mathbb{R}\alpha$ as the id. map. So the only e-value of τ on E is 1 and its min. poly. $m_\tau(T)$ divides $(T-1)^\ell$ for $\ell = \dim(E)$.

Also, if $k \geq |\Phi|$, the set $\{\beta, \tau(\beta), \dots, \tau^k(\beta)\}$ for any $\beta \in \Phi$ is a subset of Φ by invar. with $k+1$ vectors listed, so they cannot all be distinct. If $\tau^i(\beta) = \tau^j(\beta)$ for some $0 \leq i < j \leq k$ then $\tau^{j-i}(\beta) = \beta$.

For each $\beta \in \Phi$ we now have a $k_\beta \geq 1$ s.t. $\tau^{k_\beta}(\beta) = \beta$. Let $k = \text{l.c.m.}\{k_\beta \mid \beta \in \Phi\}$ so that $\forall \beta \in \Phi$, $\tau^k(\beta) = \beta$. Since $\langle \Phi \rangle = E$ this implies $\tau^k = I_E$ so τ satisfies the poly $T^k - 1$. Then $m_\tau(T)$ divides both $(T-1)^\ell$ and $(T^k - 1)$ so it divides $\text{g.c.d.}((T-1)^\ell, T^k - 1) = T - 1$ so $m_\tau(T) = T - 1$ and thus $\tau = I_E$. $\square$

Def. Say $\Phi \subset E$ is a <u>root system</u> when:

(R1) $|\Phi| < \infty$, $\langle \Phi \rangle = E$, $0 \notin \Phi$.

(R2) If $\alpha \in \Phi$ then $c\alpha \in \Phi$ iff $c = \pm 1$.

(R3) If $\alpha \in \Phi$ then $\sigma_\alpha(\Phi) = \Phi$.

(R4) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Note: If $(\cdot,\cdot)$ is rescaled by a nonzero real factor, these axioms still hold. 180

Def. For root system Φ in E let the **Weyl gp.** $W = \langle \sigma_\alpha \in GL(E) \mid \alpha \in \Phi \rangle \leq GL(E)$.

Then W acts as permutations of Φ, so Φ finite implies W is finite.

Lemma: Let Φ be a root system in E with Weyl gp. W. If $\sigma \in GL(E)$ leaves Φ invariant then $\forall \alpha \in \Phi$, $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ $\forall \alpha, \beta \in \Phi$.

Pf. $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma_\alpha(\beta) \in \Phi$ since $\sigma_\alpha(\beta) \in \Phi$, and $\sigma \sigma_\alpha(\beta) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$.

$\{\sigma(\beta) \mid \beta \in \Phi\} = \Phi$ so $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\alpha)) = -\sigma(\alpha)$ and

$\forall x \in P_\alpha$, $\sigma\sigma_\alpha\sigma^{-1}(\sigma(x)) = \sigma\sigma_\alpha(x) = \sigma(x)$
so $\sigma\sigma_\alpha\sigma^{-1}$ fixes pointwise $\sigma(P_\alpha)$, and it leaves Φ invariant as a set. By Lemma on page 177,
$\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ and so

$$\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle\sigma(\beta), \sigma(\alpha)\rangle\sigma(\alpha)$$
$$= \sigma(\beta) - \langle\beta, \alpha\rangle\sigma(\alpha)$$ from beginning

of the proof, we get $\langle\sigma(\beta), \sigma(\alpha)\rangle = \langle\beta, \alpha\rangle$. $\square$

__Def__. Let (Φ, E) and (Φ', E') be two root systems. Say they are __isomorphic__ if $\exists\ \phi : E \to E'$ vector space isomorphism (need not be an isometry) s.t. $\phi(\Phi) = \Phi'$ and $\langle\phi(\beta), \phi(\alpha)\rangle = \langle\beta, \alpha\rangle$, $\forall \alpha, \beta \in \Phi$.

Then $\sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\sigma_\alpha(\beta))$ gives an isom.