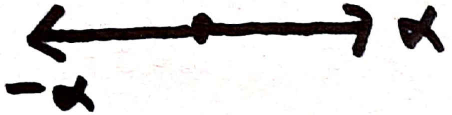
between the Weyl gp of Φ,
$W(\Phi) = \langle \sigma_\alpha \in GL(E) \mid \alpha \in \Phi \rangle$ and the Weyl gp. of Φ',
$W(\Phi') = \langle \sigma_{\phi(\alpha)} \in GL(E') \mid \alpha \in \Phi \rangle$; $\phi^{-1} \circ \sigma_{\phi(\alpha)} \circ \phi = \sigma_\alpha$
or $\sigma_{\phi(\alpha)} = \phi \circ \sigma_\alpha \circ \phi^{-1}$ for generators, so
$\forall \sigma \in W(\Phi)$, $\phi \circ \sigma \circ \phi^{-1} \in W(\Phi')$ is a gp. isom.
We can apply this to $\phi : E \to E$ s.t. $\phi(\Phi) = \Phi$
to understand $Aut(\Phi) = \{\phi : E \to E \mid \phi(\Phi) = \Phi\}$
by using the last Lemma. So $W(\Phi) \leq Aut(\Phi)$.
__Def__: For $\alpha \in \Phi$ let $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ and $\Phi^\vee = \{\alpha^\vee \in E \mid \alpha \in \Phi\}$
be the "dual" of Φ.
__Exercise__: $\Phi^\vee$ is also a root system in E and
$W(\Phi^\vee) \cong W(\Phi)$ is a natural isom.

Small rank examples: $\ell = \dim(E) = \operatorname{rank}(\Phi)$

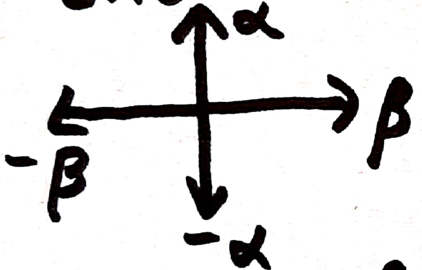
If $\ell = 1$ the only root system is $\Phi = \{\alpha, -\alpha\}$



$W = \langle \sigma_\alpha \rangle = \{I, \sigma_\alpha\} \cong \mathbb{Z}_2$. Denoted A_1

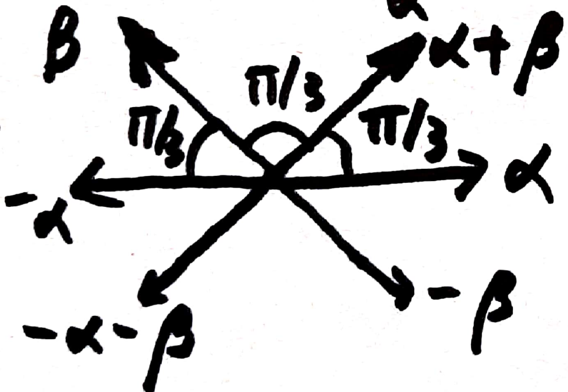
If $\ell = 2$ there are 4 possible root systems:

$A_1 \times A_1$:



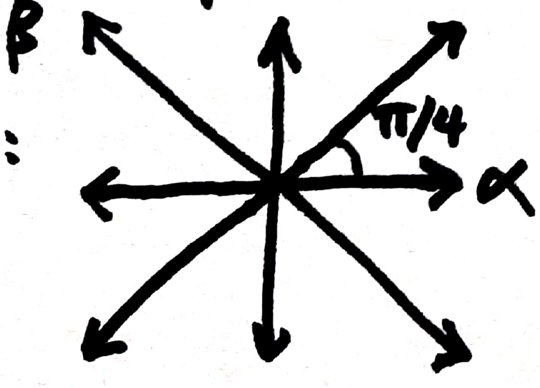
$W = \langle \sigma_\alpha, \sigma_\beta \mid 1 = \sigma_\alpha^2 = \sigma_\beta^2 = (\sigma_\alpha\sigma_\beta)^2 \rangle$
$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

A_2:



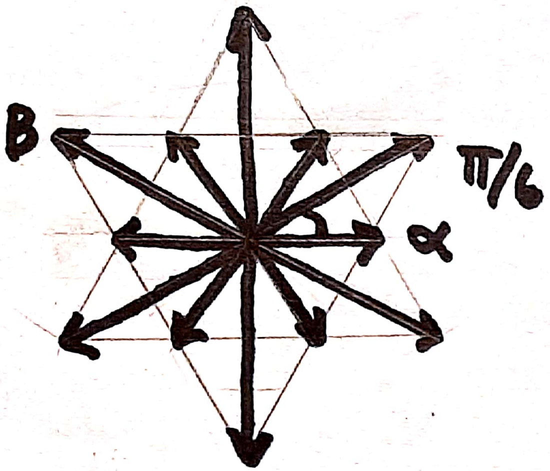
$W = \langle \sigma_\alpha, \sigma_\beta \mid 1 = \sigma_\alpha^2 = \sigma_\beta^2 = (\sigma_\alpha\sigma_\beta)^3 \rangle$
$\cong S_3 \cong D_6$ (dihedral gp)

B_2:



$W \cong \langle \sigma_\alpha, \sigma_\beta \mid 1 = \sigma_\alpha^2 = \sigma_\beta^2 = (\sigma_\alpha\sigma_\beta)^4 \rangle$
$\cong D_8$ (dihedral gp)

G_2:



$$W \cong \langle \sigma_\alpha, \sigma_\beta \mid 1 = \sigma_\alpha^2 = \sigma_\beta^2 = (\sigma_\alpha \sigma_\beta)^6 \rangle$$
$$\cong D_{12} \text{ (dihedral gp.)}$$

How does axiom (R4) limit the possible angles between roots of Φ? We know that $\|\alpha\| \cdot \|\beta\| \cos\theta = (\alpha, \beta)$ if θ is the angle between α and β. So $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\alpha\| \|\beta\| \cos\theta}{\|\alpha\|^2}$
$= \frac{2\|\beta\|}{\|\alpha\|} \cos\theta$ and $\langle \alpha, \beta \rangle = \frac{2\|\alpha\|}{\|\beta\|} \cos\theta$ as well.

So $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta \geq 0$ is an integer since $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle \in \mathbb{Z}$ are Cartan integers.

Also, $0 \le \cos^2\theta \le 1$ and $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ have the same sign as (α, β).
When $\beta = \pm\alpha$ we have $\langle \alpha, \beta \rangle = \pm 2 = \langle \beta, \alpha \rangle$ and $\theta = 0$ or π. Suppose $\beta \neq \pm\alpha$ and $\|\beta\| \geq \|\alpha\|$.
Then the options are listed in the following table:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\|\beta\|^2 / \|\alpha\|^2$	
0	0	$\pi/2$	—	$A_1 \times A_1$
1	1	$\pi/3$	1	A_2
-1	-1	$2\pi/3$	1	
1	2	$\pi/4$	2	B_2
-1	-2	$3\pi/4$	2	
1	3	$\pi/6$	3	G_2
-1	-3	$5\pi/6$	3	

<u>Lemma</u>: Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$.
If $(\alpha,\beta) > 0$ (so angle θ between them is acute) then $\alpha - \beta \in \Phi$.
If $(\alpha,\beta) < 0$ (so angle θ is obtuse) then $\alpha+\beta \in \Phi$.

<u>Pf</u>. $(\alpha,\beta) > 0$ iff $\langle \alpha,\beta\rangle > 0$. From the table, If $\langle\alpha,\beta\rangle = 1$ then either $\langle \alpha,\beta\rangle = 1$ or $\langle \beta,\alpha\rangle = 1$. If $\langle\alpha,\beta\rangle = 1$ $\sigma_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta = \alpha - \beta \in \Phi$ from (R3). If $\langle\beta,\alpha\rangle = 1$ then $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \beta - \alpha \in \Phi$ so $\alpha - \beta \in \Phi$.
If $(\alpha,\beta) < 0$ then $(\alpha, -\beta) > 0$ so from above, $\alpha + \beta = \alpha - (-\beta) \in \Phi$. $\square$

Let's look at the α-string through β for $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$, in an abstract root system Φ.

We don't have rep'n theory of $\mathfrak{sl}(2,F)$ in this situation, just the root system axioms.

Consider $\{\beta + i\alpha \in \Phi \mid i \in \mathbb{Z}\} \subseteq \Phi$. Let $0 \le r, q \in \mathbb{Z}$ be maximal s.t. $\beta - r\alpha \in \Phi$ and $\beta + q\alpha \in \Phi$.

Suppose for some $-r \le i \le q$ that $\beta + i\alpha \notin \Phi$. Then can find $-r \le p < s \le q$ such that

$\beta + p\alpha \in \Phi$, $\beta + (p+1)\alpha \notin \Phi$, $\beta + (s-1)\alpha \notin \Phi$, $\beta + s\alpha \in \Phi$,

so $\underbrace{\beta - r\alpha, \ldots, \beta + p\alpha}_{\in \Phi}, \underbrace{\beta + (p+1)\alpha}_{\notin \Phi}, \ldots, \underbrace{\beta + (s-1)\alpha}_{\notin \Phi}, \underbrace{\beta + s\alpha, \ldots, \beta + q\alpha}_{\in \Phi}$

By Lemma, $(\alpha, \beta + p\alpha) \ge 0$ and $(\alpha, \beta + s\alpha) \le 0$ so $(\alpha, \beta) + p(\alpha, \alpha) \ge 0$ and $(\alpha, \beta) + s(\alpha, \alpha) \le 0$ so $-p(\alpha, \alpha) \le (\alpha, \beta) \le -s(\alpha, \alpha)$ so $p \ge s$ contradicts $p < s$ and $(\alpha, \alpha) > 0$.

<u>Conclusion</u>: $\{\beta+i\alpha\in\Phi \mid -r\le i\le q\}$ is an unbroken string of roots, invariant under σ_α whose action reverses the order of the string.

$$\sigma_\alpha(\beta+i\alpha)=\beta+i\alpha-\langle\beta+i\alpha,\alpha\rangle\alpha$$
$$=\beta+i\alpha-\langle\beta,\alpha\rangle\alpha-i\langle\alpha,\alpha\rangle\alpha$$
$$=\beta+i\alpha-\langle\beta,\alpha\rangle\alpha-2i\alpha$$
$$=\beta-(i+\langle\beta,\alpha\rangle)\alpha=\sigma_\alpha(\beta)-i\alpha$$

For $i=q$, $\beta+q\alpha$ is the end of the string in the α direction and $\beta-r\alpha$ is the opposite end so $\sigma_\alpha(\beta+q\alpha)=\beta-r\alpha$ says

$\beta-(q+\langle\beta,\alpha\rangle)\alpha=\beta-r\alpha$ so $\langle\beta,\alpha\rangle=r-q$.

<u>Cor</u>: Root strings are of length at most 4.
(See G_2 root system diagram.)