Simple roots and Weyl group:

Def. Call $\Delta \subseteq \Phi$ a *base* of Φ if

(B1) Δ is a basis of E,

(B2) $\forall \beta \in \Phi$ can write $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ where either $0 \le k_\alpha \in \mathbb{Z}$ or $0 \ge k_\alpha \in \mathbb{Z}$.

Say $\alpha \in \Delta$ are *simple* roots. $|\Delta| = \ell = \dim(E)$.

Def. $\forall \beta \in \Phi$, the height of β w.r.t. Δ is $ht(\beta) = \sum_{\alpha \in \Delta} k_\alpha$

If $0 \le k_\alpha$ say $\beta > 0$ (is positive), and
if $0 \ge k_\alpha$ say $\beta < 0$ (is negative).

$\Phi^{\pm} = \{\beta \in \Phi \mid \pm\beta > 0\}$, $\Phi^- = -\Phi^+$, $\Phi = \Phi^+ \cup \Phi^-$.

For $\alpha, \beta \in \Phi^+$, if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.

Def. Define a partial order on E by $\mu \le \lambda$ iff $\lambda = \mu$ or $\lambda - \mu = \sum_{\alpha \in \Delta} k_\alpha \alpha$ for all $k_\alpha \ge 0$ in $\mathbb{Z}$

This is an important and useful concept, but we don't yet know that a base exists for any root system. In the rank 2 root systems we have seen, $\Delta = \{\alpha, \beta\}$ is a base, and $(\alpha, \beta) \leq 0$.

<u>Lemma</u>. If Δ is a base of Φ then $\forall \alpha \neq \beta$ in Δ we have $(\alpha, \beta) \leq 0$ and $\alpha - \beta \notin \Phi$.

<u>Pf</u>. Suppose $(\alpha, \beta) > 0$. Given $\alpha \neq \beta$ we must also have $\alpha \neq -\beta$ since otherwise $(\alpha, \beta) = (-\beta, \beta) < 0$. By last Lemma, $\alpha - \beta \in \Phi$ would contradict axiom (B2). ∎

<u>Th</u>. Φ has a base.

We will develop a method to construct all possible bases.

<u>Def</u>. $\forall\, 0 \neq \gamma \in E$ let $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$ be those roots on the side of hyperplane P_γ containing γ.

<u>Fact</u>. E is not a finite union of hyperplanes, so $\bigcup_{\alpha\in\Phi} P_\alpha \subsetneqq E$. Say $\gamma \in E - \bigcup_{\alpha\in\Phi} P_\alpha$ is <u>regular</u>, but $\gamma \in \bigcup_{\alpha\in\Phi} P_\alpha$ is <u>singular</u>. γ regular means $(\gamma,\alpha) \neq 0$, $\forall \alpha \in \Phi$, and then $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$

<u>Def</u>. Say $\alpha \in \Phi^+(\gamma)$ is <u>decomposable</u> if $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Phi^+(\gamma)$, otherwise say α is <u>indecomposable</u>.

<u>Th</u>. Let $\gamma \in E$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ, and every base of Φ is a $\Delta(\gamma)$ for some regular γ.

<u>Pf</u>. <u>Step(1)</u>: $\forall \alpha \in \Phi^+(\gamma)$, $\alpha = \sum_{\beta\in\Delta(\gamma)} k_\beta \beta$ for $0 \le k_\beta \in \mathbb{Z}$.

If not, let $\alpha \in \Phi^+(\gamma)$ be chosen so (γ,α) is minimal.

Clearly, $\alpha \notin \Delta(\gamma)$ so it is decomposible,
$\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \Phi^+(\gamma)$, and $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$
and $(\gamma, \beta_i) > 0$ so each is smaller than (γ, α). By choice
of α, each β_i is a non-neg. $\mathbb{Z}$-lin. comb. from $\Delta(\gamma)$,
so their sum is also, contradiction.

<u>Step (2)</u>: If $\alpha, \beta \in \Delta(\gamma)$ with $\alpha \neq \beta$ then $(\alpha, \beta) \leq 0$.
If $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Phi$ so either $\alpha - \beta \in \Phi^+(\gamma)$ or $\beta - \alpha \in \Phi^+(\gamma)$.
If $\alpha - \beta \in \Phi^+(\gamma)$ then $\alpha = \beta + (\alpha - \beta)$ makes α decomposable.
If $\beta - \alpha \in \Phi^+(\gamma)$ then $\beta = \alpha + (\beta - \alpha)$ makes β decomposable.

<u>Step (3)</u>: $\Delta(\gamma)$ is lin. indep.
Suppose $\sum_{\alpha \in \Delta(\gamma)} r_\alpha \alpha = 0$ for $r_\alpha \in \mathbb{R}$. Separate $r_\alpha > 0$ from those with $r_\alpha < 0$ to get

$$\varepsilon = \sum_{\alpha \in S_1} s_\alpha \alpha = \sum_{\beta \in S_2} t_\beta \beta$$

for $S_1 \cap S_2 = \phi$ subsets of $\Delta(\gamma)$, $s_\alpha > 0$, $t_\beta > 0$.

Then $(\varepsilon, \varepsilon) = \sum_{\alpha \in S_1} \sum_{\beta \in S_2} s_\alpha t_\beta (\alpha, \beta) \leq 0$ by step (2), so $\varepsilon = 0$.

Then $0=(\gamma,\varepsilon)=\sum_{\alpha\in S_1} s_\alpha(\gamma,\alpha)>0$ is a contradiction.

<u>Note</u>: The argument shows that any set of vectors on one side of a hyperplane in E with all pairs of vectors at an obtuse angle ($\theta\geq\pi/2$) must be lin. indep.

<u>Step (4)</u>: $\Delta(\gamma)$ is a base of Φ. $\Phi=\Phi^+(\gamma)\cup-\Phi^+(\gamma)$ and Step (1) give (B2). $\langle\Phi\rangle=E$ and $\langle\Delta(\gamma)\rangle_{\mathbb{Z}}\supseteq\Phi$ so $\langle\Delta(\gamma)\rangle=E$. Step (3) shows $\Delta(\gamma)$ is a basis of E, so (B1) is true.

<u>Step (5)</u>: Each base Δ of Φ is a $\Delta(\gamma)$ for some regular $\gamma\in E$. Given base Δ of Φ, find $\{\gamma\in E\,|\,(\gamma,\alpha)>0, \alpha\in\Delta\}$ is non-empty from Exercise 7 (p. 54 with hint), and pick any γ from that set. It must be regular by (B2) and $\Phi^+\subset\Phi^+(\gamma)$, $\Phi^-\subset-\Phi^+(\gamma)$ so both are "=". $\Phi^+=\Phi^+(\gamma)$ means $\Delta\subseteq\Delta(\gamma)$ are indecomp. roots. $|\Delta|=\ell=|\Delta(\gamma)|$ so $\Delta=\Delta(\gamma)$. $\square$

Def. The connected components of $E - \bigcup_{\alpha\in\Phi} P_\alpha$ are called (open) <u>Weyl chambers</u>.

Each regular $\gamma \in E$ is in exactly one Weyl chamber, denoted by $\mathcal{C}(\gamma)$. $\mathcal{C}(\gamma) = \mathcal{C}(\gamma')$ iff γ and γ' are on the same side of each hyperplane P_α, $\alpha \in \Phi$, so $\Phi^+(\gamma) = \Phi^+(\gamma')$ and $\Delta(\gamma) = \Delta(\gamma')$. This gives a bijection between Weyl chambers and bases.

Def. Let $\mathcal{C}(\Delta) = \mathcal{C}(\gamma)$ if $\Delta = \Delta(\gamma)$ and call this the <u>fundamental</u> Weyl chamber relative to Δ.

$$\mathcal{C}(\Delta) = \bigcap_{\alpha\in\Delta} \{\gamma \in E \mid (\gamma,\alpha) > 0\}$$ is an open convex set bounded by the hyperplanes P_α, $\alpha \in \Delta$.

<u>Exercise.</u> For each rank 2 root system, relative to our choice of base $\Delta = \{\alpha, \beta\}$, draw the fundamental chamber. See Fig. 1 on p. 49 for case of A_2.