**Note:** $\forall \sigma \in W$, $\sigma(\mathcal{C}(\gamma)) = \mathcal{C}(\sigma\gamma)$ for $\gamma$ regul.
W also permutes bases, $\sigma(\Delta(\gamma)) = \Delta(\sigma\gamma)$.

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Let $\Delta$ be a fixed base of $\Phi$.

**Lemma A.** If $\alpha \in \Phi^+$ with $\alpha \notin \Delta$ then $\exists \beta \in \Delta$ s.t. $\alpha - \beta \in \Phi^+$.

**Pf.** If $(\alpha, \beta) \leq 0$, $\forall \beta \in \Delta$, then $\Delta \cup \{\alpha\}$ would be indep. (See Pf of last Thm, Step (3).), impossible since $\Delta$ is a basis of $E$. So $\exists \beta \in \Delta$ s.t. $(\alpha, \beta) > 0$ so $\alpha - \beta \in \Phi$ $(\beta \neq \pm\alpha)$. If $\alpha = \sum_{\gamma \in \Delta} k_\gamma \gamma$ with all $k_\gamma \geq 0$ and some $k_\gamma > 0$ for $\gamma \neq \beta$. Then $\alpha - \beta$ is an integral lin. comb. of simple roots with at least one coeff $> 0$ so they are all $\geq 0$, so $\alpha - \beta \in \Phi^+$. $\square$

**Cor:** Each $\beta \in \Phi^+$ can be written as $\alpha_1 + \cdots + \alpha_k$ for $\alpha_i \in \Delta$ (repetitions allowed) s.t. $\alpha_1 + \cdots + \alpha_i \in \Phi^+$ for $1 \leq i \leq k$.

**Pf.** By induction on $ht(\alpha)$ using Lemma A.

**Lemma B.** If $\alpha \in \Delta$ then $\sigma_\alpha$ permutes $\Phi^+ - \{\alpha\}$.

**Pf.** Let $\beta \in \Phi^+ - \{\alpha\}$, $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$, $k_\gamma \geq 0$. $\beta \neq \pm\alpha$, so $k_\gamma > 0$ for some $\gamma \neq \alpha$. Then $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ also has coeff. $k_\gamma$ for $\gamma$ in its sum. So all its coeffs must be $\geq 0$ so $\sigma_\alpha(\beta) \in \Phi^+$. Also, $\sigma_\alpha(\beta) \neq \alpha$ since otherwise $\beta = -\alpha \in \Phi^-$. $\square$

**Cor.** For $\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$ we have $\sigma_\alpha(\delta) = \delta - \alpha$, $\forall \alpha \in \Delta$.

**Pf.** $\sigma_\alpha(\delta) = \frac{1}{2} \sum_{\beta \in \Phi^+} \sigma_\alpha(\beta) = \frac{1}{2} \sum_{\alpha \neq \beta \in \Phi^+} \sigma_\alpha(\beta) + \frac{1}{2} \sigma_\alpha(\alpha)$

$$= \frac{1}{2} \sum_{\beta \in \Phi^+ - \{\alpha\}} \beta - \frac{1}{2}\alpha = \delta - \alpha. \quad \square$$

**Lemma C.** Let $\alpha_1, \dots, \alpha_t \in \Delta$ (repeats allowed). Write $\sigma_i = \sigma_{\alpha_i} \in W$. If $\sigma_1 \cdots \sigma_{t-1}(\alpha_t) \in \Phi^-$ then for some index $1 \leq s < t$, $\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$.

Pf. Let $\beta_i = \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t)$ for $0 \leq i \leq t-2$ and $\beta_{t-1} = \alpha_t$. Since $\beta_0 < 0$ was given and $\beta_{t-1} > 0$ there is a smallest index $s$ such that $\beta_s > 0$, so $\sigma_s(\beta_s) = \beta_{s-1} < 0$. Lemma B tells us that $\sigma_s$ permutes $\Phi^+ - \{\alpha_s\}$ so $\beta_s = \alpha_s$. We know that $\forall \sigma \in W$, $\sigma_{\sigma(\alpha)} = \sigma \sigma_\alpha \sigma^{-1}$, so using $\sigma = \sigma_{s+1} \cdots \sigma_{t-1}$, $\sigma^{-1} = \sigma_{t-1} \cdots \sigma_{s+1}$ and $\sigma_\alpha = \sigma_{\alpha_t} = \sigma_t$ we get $(\sigma_{s+1} \cdots \sigma_{t-1}) \sigma_t (\sigma_{t-1} \cdots \sigma_{s+1}) = \sigma_{(\sigma_{s+1} \cdots \sigma_{t-1})\alpha_t} = \sigma_{\beta_s} = \sigma_{\alpha_s} = \sigma_s$ which means $\sigma_{s+1} \cdots \sigma_{t-1} \sigma_t = \sigma_s \sigma_{s+1} \cdots \sigma_{t-1}$ so $\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_s (\sigma_{s+1} \cdots \sigma_{t-1} \sigma_t) = \sigma_1 \cdots (\sigma_s \sigma_s) \sigma_{s+1} \cdots \sigma_{t-1}$. $\square$

**Cor.** If $\sigma_1 \cdots \sigma_t = \sigma \in W$ is an expression 198
for $\sigma$ as a product of "simple" reflections
$\sigma_i = \sigma_{\alpha_i}$, $\alpha_i \in \Delta$, with $t$ minimal, then
$\sigma(\alpha_t) < 0$.

**Pf.** Otherwise, $\sigma(\alpha_t) > 0$ says $\sigma(\alpha_t) =$
$\sigma_1 \cdots \sigma_{t-1} \sigma_t(\alpha_t) = \sigma_1 \cdots \sigma_{t-1}(-\alpha_t) > 0$ so
$\sigma_1 \cdots \sigma_{t-1}(\alpha_t) < 0$ and Lemma C gives a
shorter expression for $\sigma$ as a product of
simple reflections. □

## Weyl Group:

**Th.** Let $\Delta$ be a base of root system $\Phi$.

(a) If $\gamma \in E$ is regular then $\exists \sigma \in W$ s.t. $(\sigma(\gamma), \alpha) > 0$, $\forall \alpha \in \Delta$, so $W$ act transitively on the set of all Weyl chambers.

(b) If $\Delta'$ is another base of $\Phi$ then $\exists \sigma \in W$ s.t. $\sigma(\Delta') = \Delta$, so $W$ acts trans. on bases.

(c) $\forall \alpha \in \Phi$, $\exists \sigma \in W$ s.t. $\sigma(\alpha) \in \Delta$.

(d) $W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$, $W$ is generated by simple refl's.

(e) If $\sigma(\Delta) = \Delta$ for $\sigma \in W$ then $\sigma = 1$, so $W$ acts simply transitively on bases.

**Pf.** Let $W' = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$. We will prove (a)-(c) for $W'$ and then get $W = W'$.

Recall $\delta = \frac{1}{2}\sum_{\alpha>0} \alpha$ and choose $\sigma \in W'$ s.t. $(\sigma(\gamma),\delta)$ is maximal. $\forall \alpha \in \Delta$, $\sigma_\alpha \sigma \in W'$ so

$$(\sigma(\gamma),\delta) \geq (\sigma_\alpha\sigma(\gamma),\delta) = (\sigma(\gamma),\sigma_\alpha(\delta))$$
$$= (\sigma(\gamma),\delta-\alpha) = (\sigma(\gamma),\delta)-(\sigma(\gamma),\alpha),$$ so $(\sigma(\gamma),\alpha) \geq 0$.

$\gamma$ is regular so $(\sigma(\gamma),\alpha) \neq 0$, $\forall \alpha \in \Delta$, since otherwise $\gamma \in P_{\sigma^{-1}(\alpha)}$. Thus, $\forall \alpha \in \Delta$, $(\sigma(\gamma),\alpha) > 0$ which means $\sigma(\gamma) \in C(\Delta)$, the fundamental chamber relative to $\Delta$, and $\sigma C(\gamma) = C(\Delta)$.

(b) $W'$ permutes the Weyl chambers, which are in one-to-one correspondence with bases by $C(\gamma) = C(\Delta(\gamma))$ so $W'$ permutes the bases transitively, $\sigma C(\gamma) = C(\sigma\gamma)$ says

$$\sigma C(\Delta(\gamma)) = C(\Delta(\sigma\gamma)).$$