(c) It will be enough to prove that each $\alpha \in \Phi$ is in some base. For $\beta \neq \pm\alpha$, $P_\beta \neq P_\alpha$ so $P_\alpha \not\subseteq \bigcup_{\beta \neq \pm\alpha} P_\beta$ so $\exists \gamma \in P_\alpha$ s.t. $\gamma \notin \bigcup_{\beta \neq \pm\alpha} P_\beta$.

Choose γ' close to γ s.t. $(\gamma', \alpha) > \varepsilon > 0$ and with $|(\gamma', \beta)| > \varepsilon$, $\forall \beta \neq \pm\alpha$. Then $\alpha \in \Delta(\gamma')$.

(d) $\forall \alpha \in \Phi$ let $\sigma \in W'$ s.t. $\sigma(\alpha) = \beta \in \Delta$, so $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma \sigma_\alpha \sigma^{-1}$ so $\sigma_\alpha = \sigma^{-1} \sigma_\beta \sigma \in W'$.

This shows $W = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle = W'$.

(e) If $\sigma(\Delta) = \Delta$ for $1 \neq \sigma \in W$, write σ as a minimal length product of simple reflections as in the Cor. on page 198, $\sigma = \sigma_1 \cdots \sigma_t$. Then $\sigma(\alpha_t) < 0$ contradicts $\sigma(\Delta) = \Delta$. $\square$

<u>Def</u>. When $\sigma \in W$ is written as a product of simple roots, $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$, for $\alpha_i \in \Delta$ and t <u>minimal</u> we call that a <u>reduced</u> expression for σ and call $t = \ell(\sigma)$ the <u>length</u> of σ relative to Δ. $\ell(1) = 0$ by definition.

<u>Def</u>. $\forall \sigma \in W$, let $n(\sigma)$ be the number of $\alpha \in \Phi^+$ s.t. $\sigma(\alpha) < 0$.

<u>Lemma A</u>. $\forall \sigma \in W$, $\ell(\sigma) = n(\sigma)$.

<u>Pf</u>. By ind. on $\ell(\sigma)$. If $\ell(\sigma) = 0$ then $\sigma = 1$ so $n(\sigma) = 0$. Lemma B on page 196 gives the case $\ell(\sigma) = 1$. Assume $\ell(\tau) = n(\tau)$ for all $\tau \in W$ with $\ell(\tau) < \ell(\sigma)$. Write $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ in reduced form and let $\alpha = \alpha_t$ so $\sigma(\alpha) < 0$ by Cor. to Lemma C.

Lemma B says σ_α permutes $\Phi^+ - \{\alpha\}$ and we know $\sigma_\alpha(\alpha) = -\alpha < 0$, so $n(\sigma\sigma_\alpha) = n(\sigma) - 1$ (α is the only positive root counted in $n(\sigma)$ which is not counted in $n(\sigma\sigma_\alpha)$. $\sigma\sigma_\alpha(\alpha) = \sigma(-\alpha) > 0$.)

But $\sigma\sigma_\alpha = \sigma_1 \cdots \sigma_{t-1}\sigma_t\sigma_t = \sigma_1 \cdots \sigma_{t-1}$ has $\ell(\sigma\sigma_\alpha) = t-1 < \ell(\sigma)$, so by induction, $\ell(\sigma\sigma_\alpha) = n(\sigma\sigma_\alpha)$ says $\ell(\sigma) - 1 = n(\sigma) - 1$ so $\ell(\sigma) = n(\sigma)$. □

<u>Def</u>. A <u>fundamental</u> <u>domain</u> for W acting on E is a subset $D \subset E$ s.t. each $x \in E$ is W-conjugate to exactly one element of D. D contains one element from each W-orbit.

<u>Lemma B.</u> Let $\lambda, \mu \in \overline{C(\Delta)}$, the closure of fundamental chamber $C(\Delta)$. If $\sigma\lambda = \mu$ for some $\sigma \in W$, then σ is a product of simple reflections σ_{α_i} s.t. $\sigma_{\alpha_i}(\lambda) = \lambda$, so $\lambda = \mu$.

<u>Pf.</u> By ind. on $\ell(\sigma)$. $\ell(\sigma) = 0$ means $\sigma = 1$ so $\lambda = \mu$. Let $\ell(\sigma) > 0$, so $n(\sigma) > 0$ and σ sends some positive root to Φ^-, so σ must send some simple root to Φ^-. Say $\exists \alpha \in \Delta$ s.t. $\sigma(\alpha) < 0$. So $0 \geq (\mu, \sigma\alpha) = (\sigma^{-1}\mu, \alpha) = (\lambda, \alpha) \geq 0$ since $\lambda, \mu \in \overline{C(\Delta)}$. Then $(\lambda, \alpha) = 0$ so $\sigma_\alpha(\lambda) = \lambda$ and $(\sigma\sigma_\alpha)\lambda = \mu$. As before, $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1$ so induction gives the result for $\sigma\sigma_\alpha$, so it is true for σ. □

Irreducible root systems

Def. Say root system Φ is <u>irreducible</u> when there is no partition $\Phi = \Phi_1 \cup \Phi_2$ of Φ into orthogonal disjoint proper subsets. Otherwise, say Φ is reducible.

EX. A_1, A_2, B_2, G_2 are irreducible root systems but $A_1 \times A_1$ is reducible with $\Phi_1 = \{\pm\alpha\}$, $\Phi_2 = \{\pm\beta\}$.

Th: Let Δ be a base of root system Φ. Then Φ is irred. iff there is no partition of Δ into orthog. proper disjoint subsets.

Pf. Suppose $\Phi = \Phi_1 \cup \Phi_2$ with $(\Phi_1, \Phi_2) = 0$ so Φ is reducible. Then $\Delta = \Delta_1 \cup \Delta_2$ for $\Delta_i = \{\alpha \in \Delta \mid \alpha \in \Phi_i\}$.

If $\Delta \subset \Phi_1$ then $(\Delta, \Phi_2) = 0$ and $\mathrm{span}(\Delta) = E$ gives $(E, \Phi_2) = 0$ contradicting Φ_2 nonempty. Similarly, cannot have $\Delta \subset \Phi_2$, so Δ_1 and Δ_2 are proper subsets of Δ. Thus, Φ reducible implies Δ is reducible.

Suppose Φ is irred. but $\Delta = \Delta_1 \cup \Delta_2$ with $(\Delta_1, \Delta_2) = 0$ and $\emptyset \neq \Delta_i \subsetneq \Delta$ disjoint. $\forall \alpha \in \Phi$ $\exists \sigma \in W$ s.t. $\sigma(\alpha) \in \Delta$. Let $\Phi_i = \{\alpha \in \Phi \mid \sigma(\alpha) \in \Delta_i$ for some $\sigma \in W\}$ so $\Phi = \Phi_1 \cup \Phi_2$. Note that if $(\alpha, \beta) = 0$ then $\sigma_\alpha(\beta) = \beta$ and $\sigma_\beta(\alpha) = \alpha$ so $\forall \gamma \in E$,

$$\sigma_\alpha \sigma_\beta(\gamma) = \sigma_\alpha(\gamma - \langle \gamma, \beta \rangle \beta) = \gamma - \langle \gamma, \alpha \rangle \alpha - \langle \gamma, \beta \rangle \beta,$$

$$\sigma_\beta \sigma_\alpha(\gamma) = \sigma_\beta(\gamma - \langle \gamma, \alpha \rangle \alpha) = \gamma - \langle \gamma, \beta \rangle \beta - \langle \gamma, \alpha \rangle \alpha$$

so $\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$.

$W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$ so $\alpha \in \Phi_i$ means $\alpha \in W(\Delta_i)$
and only simple reflections σ_β for $\beta \in \Delta_i$ can move simple roots in Δ_i, giving roots in $\mathrm{span}(\Delta_i)$.
Let $E_i = \mathrm{span}(\Delta_i)$, so $\Phi_i \subset E_i$ and $(E_1, E_2) = 0$ so $(\Phi_1, \Phi_2) = 0$. $\Delta_i \subset \Phi_i$ is nonempty, so Φ is reducible, contradiction. $\square$

<u>Lemma A</u>. Let Φ be irred. Then w.r.t. partial ordering $\leq$, there is a unique maximal root β. Thus, $\forall \beta \neq \alpha \in \Phi$, $ht(\alpha) < ht(\beta)$, and $\forall \alpha \in \Delta, (\beta, \alpha) \geq 0$.
Also, $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with all $k_\alpha > 0$.