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<u>Pf</u>. Let $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ be max. w.r.t. $\leq$.

Certainly $0 < \beta \in \Phi^+$ since $\Phi^- < \Phi^+$. Let $\Delta_1 = \{\alpha \in \Delta \mid k_\alpha > 0\}$ and $\Delta_2 = \{\alpha \in \Delta \mid k_\alpha = 0\}$, so $\Delta = \Delta_1 \cup \Delta_2$ is a disjoint union. Suppose $\Delta_2 \neq \phi$.

Then $\forall \alpha \in \Delta_2$, $(\alpha, \beta) \leq 0$. Φ irred. implies $\exists \alpha \in \Delta_2$ $\exists \alpha' \in \Delta_1$ s.t. $(\alpha, \alpha') < 0$ so $(\alpha, \beta) = \sum_{\gamma \in \Delta_1} k_\gamma (\alpha, \gamma) < 0$ since $(\alpha, \gamma) \leq 0$, $\forall \gamma \in \Delta_1$ and $(\alpha, \alpha') < 0$.

But then $\beta + \alpha \in \Phi$ contradicts max. of β.

So $\Delta_2 = \phi$ and all $k_\alpha > 0$, $\Delta_1 = \Delta$.

Thus, $\forall \alpha \in \Delta$, $(\alpha, \beta) \geq 0$ and $\exists \alpha \in \Delta$, $(\alpha, \beta) > 0$ since $(\Delta, \beta) = 0 \Rightarrow (E, \beta) = 0 \Rightarrow \beta = 0$.

To prove uniqueness of max. root, let β' be
another max. root, $\beta' = \sum_{\alpha\in\Delta} k'_\alpha \alpha$ with all $k'_\alpha > 0$
and $\exists \alpha' \in \Delta$ s.t. $(\alpha', \beta') > 0$ and $\forall \alpha \in \Delta, (\alpha, \beta') \geq 0$.
Then $(\beta, \beta') = \sum_{\alpha\in\Delta} k_\alpha (\alpha, \beta') > 0$ so $\beta - \beta' \in \Phi \cup \{0\}$.

If $\beta - \beta' \in \Phi$ either $\beta - \beta' \in \Phi^+$ so $\beta > \beta'$ or else
$\beta - \beta' \in \Phi^-$ so $\beta' - \beta \in \Phi^+$ so $\beta' > \beta$. Either way
contradicts max. of β or β', so $\beta = \beta'$. $\square$

__Lemma B.__ Let Φ be irred. Then W acts irreducibly on E, and $\forall \alpha \in \Phi$, $\mathrm{span}(W\alpha) = E$.

__Pf.__ $\mathrm{span}(W\alpha) \leq E$ is a W-invar. subspace of E, non-trivial since it contains α. Let $0 \neq E' \leq E$ be a W-invar subspace, $E'' = (E')^\perp$ its orthog. complement.
Then E'' is also W-invar. and $E = E' \oplus E''$.

$\forall\, x' \in E',\ y'' \in E'',\ \sigma \in W,\ (x', \sigma(y'')) = (\sigma^{-1}(x'), y'')$
$= 0$ since $\sigma^{-1}(x') \in E'$.

On page 45 of Humphreys, Exercise 1, you are asked to prove that if reflection $\sigma_\alpha(E') \subseteq E'$ then either $\alpha \in E'$ or $E' \subset P_\alpha$. Pf. Suppose $\sigma_\alpha(E') \subseteq E'$ but $\alpha \notin E'$. Then $\forall\, x' \in E'$, $\sigma_\alpha(x') = x' - \langle x', \alpha \rangle \alpha \in E'$ gives $\langle x', \alpha\rangle \alpha \in E'$ so $\langle x', \alpha \rangle = 0$ so $x' \in P_\alpha$ so $E' \subset P_\alpha$ and $\alpha \in (E')^\perp$. □

Applying this to $E = E' \perp E''$ (orthog. dir. sum), get $\alpha \notin E' \Rightarrow \alpha \in (E')^\perp = E''$. $\forall\, \alpha \in \Phi$, either $\alpha \in E'$ or $\alpha \in E''$. $\Phi = \Phi' \cup \Phi''$ for $\Phi' = \{\alpha \in \Phi \mid \alpha \in E'\}$, $\Phi'' = \{\alpha \in \Phi \mid \alpha \in E''\}$ would be an orthog. partition of irred. Φ. Can't have both non-empty, so $E = E'$. □

Lemma C. Let Φ be irred. Then at most two p211 root lengths occur in Φ, and all roots of a given length are in same W orbit.

Pf. $\forall \alpha, \beta \in \Phi$, $W\alpha$ spans E so $(W\alpha, \beta) = 0$ is not possible, $\exists \sigma \in W$ s.t. $(\sigma(\alpha), \beta) \neq 0$. If $(\alpha, \beta) \neq 0$ then possible values of $(\beta,\beta)/(\alpha,\alpha) = \|\beta\|^2/\|\alpha\|^2$ are $1, 2, 3, 1/2, 1/3$ from table on page 185 of these notes. $\|\alpha\| = \|\sigma(\alpha)\|$, $\forall \sigma \in W$, and for some $\sigma \in W$, $(\sigma(\alpha), \beta) \neq 0$ so $\|\beta\|^2/\|\sigma(\alpha)\|^2 \in \{1, 2, 3, 1/2, 1/3\}$. If $\exists\, \alpha, \beta, \gamma \in \Phi$ with three distinct lengths, $\|\alpha\| < \|\beta\| < \|\gamma\|$ and ratios $\|\beta\|^2/\|\alpha\|^2, \|\gamma\|^2/\|\beta\|^2 \in \{2, 3, \quad\}$ then $\|\gamma\|^2/\|\alpha\|^2 = \frac{3}{2}$ would not be a possible ratio.

Suppose $\|\alpha\| = \|\beta\|$ for $(\alpha,\beta) \neq 0$, if necessary
after replacement of α by $\sigma(\alpha)$ for some $\sigma \in W$.
If $\alpha = \beta$ there is nothing more to prove, so suppose
$\alpha \neq \beta$. By Table on p. 185, $\langle \alpha,\beta \rangle = \langle \beta,\alpha \rangle = \pm 1$.
If $\langle \alpha,\beta \rangle = -1$, then replace β by $-\beta = \sigma_\beta(\beta)$ so
$\langle \alpha,\beta \rangle = 1$. Compute: $\sigma_\alpha \sigma_\beta \sigma_\alpha(\beta) = \sigma_\alpha \sigma_\beta(\beta - \alpha) =$
$\sigma_\alpha(-\beta - \alpha + \beta) = \sigma_\alpha(-\alpha) = \alpha$ so α and β are in the
same W orbit. $\square$

For Φ irred with two distinct root lengths,
they are called <u>long</u> or <u>short</u> roots, accordingly.
A standard normalization is to say the
long roots have $\|\alpha\|^2 = 2$.

<u>Lemma D.</u> Let Φ be irred. with two distinct root lengths. Then the max. root is long.

<u>Pf.</u> Let β be the max. root and $\alpha \in \Phi$ arbitrary. Want to show $(\beta,\beta) \geq (\alpha,\alpha)$. We can replace α by W conjugate $\sigma(\alpha) \in \overline{C(\Delta)}$ closure of fund. chamber w.r.t. Δ. Since β is max. root, $\beta \geq \sigma(\alpha)$,

$$\beta - \sigma(\alpha) = \sum_{\delta \in \Delta} k_\delta \delta \text{ for } k_\delta \geq 0, \text{ so } \forall \gamma \in \overline{C(\Delta)},$$

$$(\gamma, \beta - \sigma(\alpha)) = \sum_{\delta \in \Delta} k_\delta (\gamma, \delta) \geq 0 \text{ so } (\gamma, \beta) \geq (\gamma, \sigma(\alpha)).$$

From Lemma A, $\beta \in \overline{C(\Delta)}$, and $\sigma(\alpha) \in \overline{C(\Delta)}$, so using $\gamma = \beta$ and then $\gamma = \sigma(\alpha)$, get

$$(\beta,\beta) \geq (\beta, \sigma(\alpha)) \geq (\sigma(\alpha), \sigma(\alpha)) = (\alpha,\alpha). \quad \square$$