Classification: Cartan Matrix of Φ:

Notations: Φ root system, W Weyl group of Φ, $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ ordered base of Φ, $\ell = \dim(E)$, E Euclidean space containing Φ with inner prod. $(\cdot, \cdot)$.

<u>Def</u>. The Cartan matrix of Φ is $[\langle \alpha_i, \alpha_j \rangle]$ made from the Cartan integers, $\langle \alpha_i, \alpha_j \rangle = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ so $\langle \alpha_i, \alpha_i \rangle = 2$, $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$, $\langle \alpha_i, \alpha_j \rangle = 0$ iff $(\alpha_i, \alpha_j) = 0$ iff $\langle \alpha_j, \alpha_i \rangle = 0$.

<u>Ex.</u> $A_1 \times A_1 : \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A_2 : \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B_2 : \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$, $G_2 : \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$, $A_1 : [2]$.

<u>Note</u>: $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$ iff $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$
means $\|\alpha_i\| = \|\alpha_j\|$, but $|\langle \alpha_i, \alpha_j \rangle| > |\langle \alpha_j, \alpha_i \rangle| > 0$
means $\dfrac{2|(\alpha_i, \alpha_j)|}{(\alpha_j, \alpha_j)} > \dfrac{2|(\alpha_j, \alpha_i)|}{(\alpha_i, \alpha_i)}$ iff $\|\alpha_i\| > \|\alpha_j\|$.

So in type B_2, $\langle \alpha_1, \alpha_2 \rangle = -2$ and $\langle \alpha_2, \alpha_1 \rangle = -1$
means $\|\alpha_1\| > \|\alpha_2\|$ so α_1 is long, α_2 is short.
In type G_2, $\langle \alpha_1, \alpha_2 \rangle = -1$ and $\langle \alpha_2, \alpha_1 \rangle = -3$
so $\|\alpha_2\| > \|\alpha_1\|$ and α_2 is long, α_1 is short.
These are Humphreys' choices of the order of simple roots, made to match older literature.

<u>Note</u>: Choice of Δ does not affect the Cartan matrix (up to permutation of rows/columns) since W acts transitively on bases. Also, the Cartan matrix is invertible since $[(\alpha_i, \alpha_j)]$ is invertible matrix of the inner product.

<u>Prop</u>. Let $\Phi' \subset E'$ be another root system with base $\Delta' = \{\alpha_1', \dots, \alpha_\ell'\}$ s.t. $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle$ for $1 \le i, j \le \ell$. Then map $\Delta \to \Delta'$ by $\alpha_i \mapsto \alpha_i'$ extends uniquely to an isom. $\phi : E \to E'$ s.t. $\phi(\Phi) = \Phi'$ and $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$, $\forall \alpha, \beta \in \Phi$. The Cartan matrix of Φ determines Φ up to isomorphism.

<u>Pf</u>. Since Δ is a basis of E and Δ' is a basis of E', $\phi : E \to E'$ vector space isom. is uniquely determined

by $\phi(\alpha_i)=\alpha_i'$, $1\le i\le l$. $\forall \alpha,\beta\in\Delta$ we have
$\sigma_{\phi(\alpha)}(\phi(\beta)) = \sigma_{\alpha'}(\beta') = \beta' - \langle \beta',\alpha'\rangle\alpha' = \phi(\beta)-\langle\beta,\alpha\rangle\phi(\alpha)$
$= \phi(\beta-\langle\beta,\alpha\rangle\alpha) = \phi(\sigma_\alpha(\beta))$. It means that the
diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\sigma_\alpha"] & E' \arrow[d, "\sigma_{\phi(\alpha)}"] \\ E \arrow[r, "\phi"] & E' \end{tikzcd}$$

commutes $\forall \alpha\in\Delta$.
The Weyl groups
$W=\langle\sigma_\alpha \mid \alpha\in\Delta\}$ and
$W'=\langle\sigma_{\alpha'} \mid \alpha'\in\Delta'\}$
are isomorphic by conjugation by ϕ. From above,
$\phi^{-1}\circ\sigma_{\phi(\alpha)}\circ\phi = \sigma_\alpha$ so $\sigma_{\phi(\alpha)} = \phi\circ\sigma_\alpha\circ\phi^{-1}$ so $\forall\sigma\in W$
$\phi\circ\sigma\circ\phi^{-1}\in W'$ defines an isomorphism from W
to W' sending W generators σ_α, $\alpha\in\Delta$, to W'
generators $\sigma_{\phi(\alpha)}$, $\phi(\alpha)\in\Delta'$. Since each $\beta\in\Phi$ is
W-conjugate to a simple root, $\beta=\sigma(\alpha)$, get

$\phi(\beta) = (\phi \circ \sigma \circ \phi^{-1})(\phi(\alpha)) \in \Phi'$, and ϕ bijective implies $\phi(\Phi) = \Phi'$. Finally, $\forall \alpha, \beta \in \Phi$, we have $\sigma_{\phi(\beta)}(\phi(\alpha)) = \phi(\sigma_\beta(\alpha))$ so $\phi(\alpha) - \langle \phi(\alpha), \phi(\beta) \rangle \phi(\beta) = \phi(\alpha) - \langle \alpha, \beta \rangle \phi(\beta)$ gives $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$. $\square$

Algorithm to get Φ from $[\langle \alpha_i, \alpha_j \rangle]$:
Enough to get Φ^+. Have $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ all roots with height 1. For $1 \le i \ne j \le \ell$, find the α_j-string through α_i; $\alpha_i, \alpha_i + \alpha_j, \ldots, \alpha_i + q\alpha_j$ since $\alpha_i - \alpha_j \notin \Phi$ so $r = 0$, $\langle \alpha_i, \alpha_j \rangle = r - q = -q$. If $q > 0$, $\alpha_i + \alpha_j \in \Phi$ and $ht(\alpha_i + \alpha_j) = 2$ and all

roots of height 2 are obtained that way.
Then you will have all Cartan integers $\langle \alpha, \alpha_j \rangle$ for $ht(\alpha) = 2$, and you will know the α_j-string through α; $\alpha - r\alpha_j, \dots, \alpha, \dots, \alpha + q\alpha_j$ for $\langle \alpha, \alpha_j \rangle = r - q$. Knowing r from roots of lower height and knowing $r - q = \langle \alpha, \alpha_j \rangle$ gives q. If $q > 0$, get roots of height 3, etc. All $\alpha \in \Phi^+$ are eventually obtained.

<u>Alternative</u>: After finding α_j-string through positive α_i, apply W to get more roots, positive and negative. For all pairs of distinct roots, one simple (α_j), one found (α), repeat process until no new roots are found.

Ex: Find all positive roots Φ^+ for type A3 where the Cartan matrix is $A = [a_{ij}] = [\langle \alpha_i, \alpha_j \rangle]$

$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$

$\langle \alpha_1, \alpha_2 \rangle = -1$ so $\alpha_1 + \alpha_2 \in \Phi^+$

$\langle \alpha_2, \alpha_3 \rangle = -1$ so $\alpha_2 + \alpha_3 \in \Phi^+$

$\langle \alpha_1, \alpha_3 \rangle = 0$ so $\alpha_1 + \alpha_3 \notin \Phi$.

$\langle \alpha_1 + \alpha_2, \alpha_3 \rangle = \langle \alpha_1, \alpha_3 \rangle + \langle \alpha_2, \alpha_3 \rangle = -1$ so

$\alpha_1 + \alpha_2 + \alpha_3 \in \Phi$

$\langle \alpha_1 + \alpha_2, \alpha_1 \rangle = 2 - 1 = 1$. $\alpha_2, \alpha_1 + \alpha_2$ is the α_1-string through $\alpha_1 + \alpha_2$ with $r = 1$, $q = 0$.

$\langle \alpha_1 + \alpha_2, \alpha_2 \rangle = -1 + 2 = 1$. $\alpha_1, \alpha_1 + \alpha_2$ is the α_2-string through $\alpha_1 + \alpha_2$ with $r = 1$, $q = 0$.

So far Φ^+ contains $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$. Look for $ht(\alpha)=4$ roots:

$\langle \alpha_1+\alpha_2+\alpha_3, \alpha_1 \rangle = 2-1+0=1$ so α_1-string through $\alpha_1+\alpha_2+\alpha_3$ is $\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3$ with $r=1, q=0$

$\langle \alpha_1+\alpha_2+\alpha_3, \alpha_2 \rangle = -1+2-1=0$ so $r=0=q$.

$\langle \alpha_1+\alpha_2+\alpha_3, \alpha_3 \rangle = 0-1+2=1$. $\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3$ is the α_3-string through $\alpha_1+\alpha_2+\alpha_3$ with $r=1, q=0$.

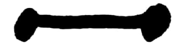
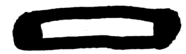
Conclusion, $\alpha_1+\alpha_2+\alpha_3$ is the highest root, and the 6 positive roots listed above are all of Φ^+.

Count dimensions of root spaces and add dim of max. toral subalg. H; get 12 roots, α, each L_α has $\dim(L_\alpha)=1$, $\dim(H)=\ell=3$, so $\dim(L)=15=\dim(\mathfrak{sl}(4,F))$.

Coxeter graphs and Dynkin diagrams: L222

For $\alpha \neq \beta$ in Φ^+, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.

__Def.__ The Coxeter graph of Φ is the graph with ℓ vertices, and having $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges joining vertex i with vertex j for $1 \leq i \neq j \leq \ell$.

__Ex:__ A_1:  $A_1 \times A_1$:  A_2:  B_2: 

G_2:  A_3: 

When $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \in \{0, 1\}$, all roots have the same length and the Coxeter graph determines the Cartan matrix. But if some values are 2 or 3, the Coxeter graph does not tell which simple roots are long and which are short.

When a double or triple edge occurs in $\mathrm{Cox}(\Phi)$, we can add an arrow pointing to the shorter root. That result is the Dynkin diagram of Φ, for example, B_2:  α_1 α_2 and

G_2:  α_1 α_2

Conversely, given Dynkin diagram  with vertices labelled left to right, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the Cartan matrix would be

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$\|\alpha_1\| = \|\alpha_2\| > \|\alpha_3\| = \|\alpha_4\|$ so

$$|\langle \alpha_2, \alpha_3 \rangle| > |\langle \alpha_3, \alpha_2 \rangle|$$

and $\|\alpha_2\|^2 / \|\alpha_3\|^2 = 2$

This type is called F_4.

Irreducible components:

Φ is irred. iff Φ (and Δ) cannot be partitioned into proper orthogonal subsets, so this means $\mathrm{Cox}(\Phi)$ is connected. Generally, if $\Delta = \Delta_1 \cup \dots \cup \Delta_t$ is a partition of Δ into mutually orthog. subsets and $E_i = \mathrm{span}(\Delta_i)$, then $E = E_1 \oplus \dots \oplus E_t$ is an orthog. direct sum and $\Phi_i = \Phi \cap \mathrm{span}(\Delta_i) \subset E_i$, $\Phi = \Phi_1 \cup \dots \cup \Phi_t$, each Φ_i is a root system with Weyl group $W_i = \langle \sigma_\alpha \mid \alpha \in \Delta_i \rangle \leq W$. $\mathrm{Cox}(\Phi) =$ disjoint union of $\mathrm{Cox}(\Phi_i)$, same for Dynkin diag.

__Prop.__ Φ decomposes uniquely as the union of irred. root systems $\Phi_i \subseteq E_i \subseteq E$ s.t. $E = E_1 \oplus \dots \oplus E_t$ is an orthog. direct sum.

Classification Thm. It suffices to classify all irred. root systems, that is, all connected Dynkin diagrams.

If Φ is an irred. root system of rank ℓ, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, then its Dynkin diagram is one of those listed in Humphreys, page 58, of type A_ℓ $(\ell \geq 1)$, B_ℓ $(\ell \geq 2)$, C_ℓ $(\ell \geq 3)$, D_ℓ $(\ell \geq 4)$, E_6, E_7, E_8, F_4, G_2, with vertices labelled $1 \leq i \leq \ell$ corresponding to $\alpha_i \in \Delta$.

The restrictions on ℓ for A_ℓ - D_ℓ avoid duplications. The corresponding Cartan matrices are listed in Table 1 on page 59.