Note: The difference between B_ℓ and C_ℓ is that B_ℓ has $\ell-1$ long simple roots and 1 short, but C_ℓ has $\ell-1$ short simple roots and 1 long.

Pf. First classify all possible Coxeter diagrams and then see what Dynkin diagrams result. Study finite sets of unit vectors whose pairwise angles are given by the $\mathrm{Cox}(\Phi)$.

Assume: E is a Euclidean space, $V = \{\varepsilon_1, \dots, \varepsilon_n\}$ a set of n lin. indep. unit vectors s.t. $(\varepsilon_i, \varepsilon_j) \le 0$ for $1 \le i \ne j \le n$ and $4(\varepsilon_i, \varepsilon_j)^2 \in \{0, 1, 2, 3\}$ for $i \ne j$. Call such a set "admissible". For example, the simple roots in a base, each made a unit vector, $\alpha_i / \|\alpha_i\| = \varepsilon_i$ would be admissible.

Let graph Γ of $\mathcal{V}$ be a Coxeter graph with n vertices, with $4(\varepsilon_i, \varepsilon_j)^2$ edges between vertex i and vertex j. Find all connected graphs of admissible sets in steps as follows. At first, don't assume Γ is connected.

<u>Step 1</u>: If some ε_i from $\mathcal{V}$ are deleted, the remaining set $\mathcal{V}'$ is still admissible, and the graph Γ' of $\mathcal{V}'$ is obtained from Γ by deleting the vertices of the deleted ε_i and all edges incident to those vertices. <u>Pf</u>. Clear.

<u>Step 2</u>: The number of pairs of vertices in Γ connected by at least one edge is strictly less than n. <u>Pf</u>. Let $\varepsilon = \sum_{i=1}^{n} \varepsilon_i \neq 0$ since $\mathcal{V}$ is indep.

So $0 < (\varepsilon, \varepsilon) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\varepsilon_i, \varepsilon_j) = n + 2 \sum_{1 \le i < j \le n} (\varepsilon_i, \varepsilon_j)$.

Suppose $i < j$ is a pair of vertices joined by at least one edge in Γ_1, so $(\varepsilon_i, \varepsilon_j) < 0$ and $4(\varepsilon_i, \varepsilon_j)^2 \in \{1, 2, 3\}$ so $2(\varepsilon_i, \varepsilon_j) \in \{-1, -\sqrt{2}, -\sqrt{3}\} \le -1$. The number of such pairs is at most $n-1$, since otherwise $0 < (\varepsilon, \varepsilon)$ is violated.

<u>Step 3</u>: Γ contains no cycles. <u>Pf</u>. A cycle in Γ would be a subgraph Γ' of an admissible subset $\mathcal{U}' \subseteq \mathcal{U}$ with m vertices and at least m edges, violating the result of Step 2 for Γ'.

<u>Step 4</u>: No more than 3 edges can be incident to a vertex of Γ. <u>Pf</u>. Let $\varepsilon \in \mathcal{U}$ and $\eta_1, \dots, \eta_k \in \mathcal{U}$ be connected to ε by at least 1 edge, so $(\varepsilon, \eta_i) < 0$

and $\varepsilon, \eta_1, \dots, \eta_k$ are all distinct. By Step 3, for $1 \le i \ne j \le k$, $(\eta_i, \eta_j) = 0$, else get a cycle. U is indep. so $\{\varepsilon, \eta_1, \dots, \eta_k\}$ is indep. and has span of dim. $k+1$, while span $\{\eta_1, \dots, \eta_k\}$ has dim. k.

The orthog. complement of $\langle \eta_1, \dots, \eta_k \rangle$ in $\langle \varepsilon, \eta_1, \dots, \eta_k \rangle$ is 1-dim'l, so $\exists\, \eta_0 \in \langle \varepsilon, \eta_1, \dots, \eta_k \rangle$ s.t. $\|\eta_0\| = 1$ and $(\eta_0, \eta_i) = 0$ for $1 \le i \le k$, but $(\varepsilon, \eta_0) \ne 0$. Then $\{\eta_0, \eta_1, \dots, \eta_k\}$ is an orthonormal basis of $\langle \varepsilon, \eta_1, \dots, \eta_k \rangle$ so $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\eta_i$ and $1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2$.

But $(\varepsilon, \eta_0) \ne 0$ so $\sum_{i=1}^{k} (\varepsilon, \eta_i)^2 < 1$, $\sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 < 4$ and $4(\varepsilon, \eta_i)^2$ is the number of edges joining ε to η_i in Γ.

<u>Step 5</u>: The only connected graph Γ of admiss. $\mathcal{U}$ containing a triple edge is  G_2.

<u>Pf.</u> From Step 4, no other vertices can be connected to either of those two.

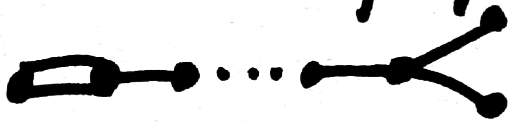
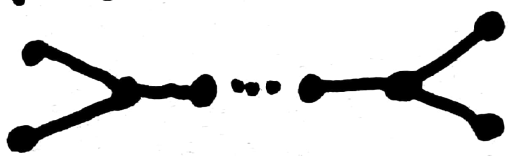
<u>Step 6</u>: Let $\{\varepsilon_1, \dots, \varepsilon_k\} \subset \mathcal{U}$ have subgraph  (a simple chain) in Γ. If $\mathcal{U}' = (\mathcal{U} - \{\varepsilon_1, \dots, \varepsilon_k\}) \cup \{\varepsilon\}$ for $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$ then $\mathcal{U}'$ is admissible. (Graph Γ' of $\mathcal{U}'$ is obtained from Γ by shrinking the simple chain to a point, and any edges connected to the simple chain are connected to that point.

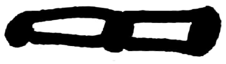
<u>Pf.</u> Clearly $\mathcal{U}'$ is indep. We are assuming (simple chain) that $2(\varepsilon_i, \varepsilon_{i+1}) = -1$ for $1 \le i \le k-1$, so

$$(\varepsilon,\varepsilon) = k + 2\sum_{1\le i<j\le k}(\varepsilon_i,\varepsilon_j) = k + 2\sum_{i=1}^{k-1}(\varepsilon_i,\varepsilon_{i+1}) = k-(k-1) = 1$$ [23]

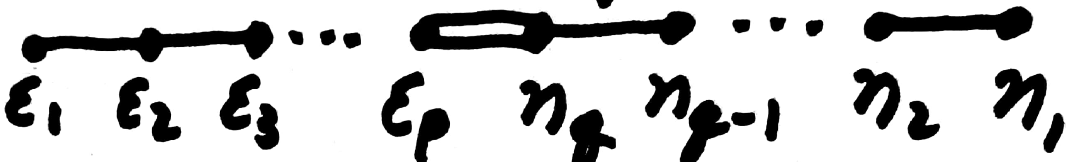
making ε a unit vector. For any $\eta \in \mathcal{U} - \{\varepsilon_1, \dots, \varepsilon_k\}$, η can be connected to at most one of $\varepsilon_1, \dots, \varepsilon_k$, else have a cycle in Γ. So $(\eta,\varepsilon) = 0$ or else $(\eta,\varepsilon) = (\eta,\varepsilon_i)$ for the $1 \le i \le k$ s.t. η is connected to ε_i. In either case, $4(\eta,\varepsilon)^2 \in \{0,1,2,3\}$ as required for $\mathcal{U}'$ to be admissible.

__Step 7.__ Γ contains no subgraph of the form:

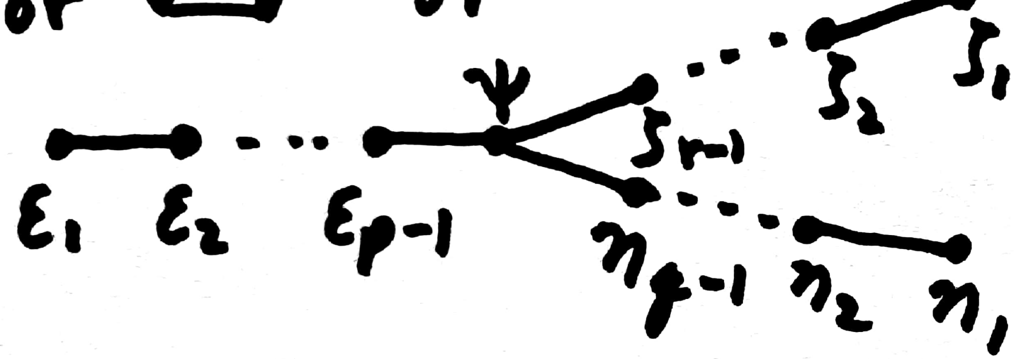
 or  or .

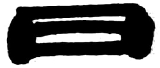
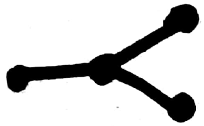
__Pf.__ If Γ contained one of these subgraphs, it would be the graph of an admissible set by Step 1. Then Step 6 would let us replace the simple chain in the middle by a single vertex, giving graphs  or  or  contradicting Step 4.

Step 8. Any connected graph Γ of admissible set $\mathfrak{V}$ has one of the following forms:

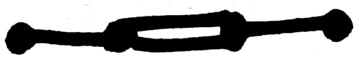
 or 

or  or



Pf. Only option for a triple edge is  G_2 by Step 5. A connected graph with more than one double edge contains a subgraph  ruled out in Step 7. So at most one double edge occurs. Similarly, Γ cannot have both a double edge and a triple branch  so only chains (simple) on either side of a double edge are possible.

If Γ has only single edges then it is either a simple chain or has (at most) one branching node.

<u>Step 9</u>. The only connected Γ with a double edge is either F_4:  or  B_n or C_n type Coxeter graphs.

<u>Pf</u>. Let notation for vertices be as in Step 8, and let $\varepsilon = \sum_{i=1}^{p} i\,\varepsilon_i$, $\eta = \sum_{i=1}^{q} i\,\eta_i$. We are assuming

$2(\varepsilon_i, \varepsilon_{i+1}) = -1$ for $1 \le i \le p-1$ and

$2(\eta_i, \eta_{i+1}) = -1$ for $1 \le i \le q-1$ and $4(\varepsilon_p, \eta_q)^2 = 2$,

but all other pairings are 0. Then,

$$(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p^2 - \sum_{i=1}^{p-1} i = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}$$

and similarly, $(\eta, \eta) = \frac{q(q+1)}{2}$. Also $(\varepsilon, \eta) = (p\varepsilon_p, q\eta_q) = pq(\varepsilon_p, \eta_q)$ so $(\varepsilon, \eta)^2 = p^2q^2/2$. The Cauchy-Schwarz ineq. for lin. indep. vectors ε and η says $(\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$, that is,

$$\frac{p^2q^2}{2} < \frac{p(p+1)\,q(q+1)}{4} \text{ so } 2pq < (p+1)(q+1)$$

so $pq - p - q < 1$ so $pq - p - q + 1 < 2$ so $(p-1)(q-1) < 2$. The only options are: $p = 2 = q$ (F_4), $p = 1$, $q \geq 1$ or $p \geq 1$, $q = 1$ (symmetric) the B_n or C_n type Coxeter graph.