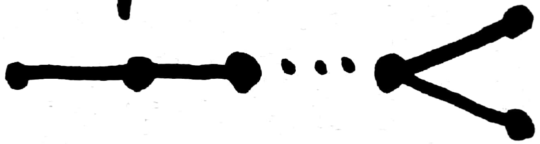
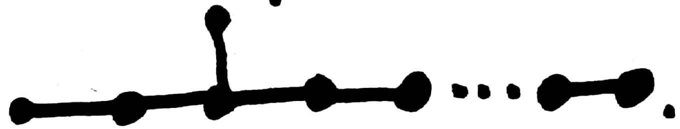
<u>Step 10</u>: The only connected Γ with a triple branching node is either type D_4: L235



or E_6, E_7, E_8:



<u>Pf</u>. With notation for vertices as in Step 8, let $\varepsilon = \sum_{i=1}^{p-1} i\varepsilon_i$, $\eta = \sum_{i=1}^{q-1} i\eta_i$, $\zeta = \sum_{i=1}^{r-1} i\zeta_i$, so ε, η, ζ are mutually orthogonal, $\{\varepsilon, \eta, \zeta, \psi\}$ is indep. Let the angle between ψ and ε be θ_1, between ψ and η be θ_2 and between ψ and ζ be θ_3. We know $\langle \psi, \varepsilon \rangle \langle \varepsilon, \psi \rangle = 4\cos^2\theta_1$

$$= \frac{2(\psi,\varepsilon)}{(\varepsilon,\varepsilon)} \frac{2(\varepsilon,\psi)}{(\psi,\psi)} = \frac{4(\psi,\varepsilon)^2}{(\varepsilon,\varepsilon)}$$

since $\|\psi\| = 1$, and similarly for η and ζ,

$\langle\psi,\eta\rangle\langle\eta,\psi\rangle = 4\cos^2\theta_2$ and $\langle\psi,\zeta\rangle\langle\zeta,\psi\rangle = 4\cos^2\theta_3$. In Step 4 we had a vertex connected by at least one edge to a list of other vertices which were mutually orthog. We have that situation now, with ψ the central vertex connected with ε, η, ζ. As in Step 4, we make the argument that $\exists\, \psi_0 \in \langle \varepsilon, \eta, \zeta, \psi\rangle$ s.t. $(\psi_0, \varepsilon) = (\psi_0, \eta) = (\psi_0, \zeta) = 0$ and $\|\psi_0\| = 1$ so $\{\varepsilon, \eta, \zeta, \psi_0\}$ is an orthogonal basis of $\langle \varepsilon, \eta, \zeta, \psi\rangle$. We can then write

$$\psi = \frac{(\psi,\varepsilon)}{(\varepsilon,\varepsilon)}\varepsilon + \frac{(\psi,\eta)}{(\eta,\eta)}\eta + \frac{(\psi,\zeta)}{(\zeta,\zeta)}\zeta + (\psi,\psi_0)\psi_0 \quad \text{so}$$

$$1 = (\psi,\psi) = \frac{(\psi,\varepsilon)^2}{(\varepsilon,\varepsilon)^2}(\varepsilon,\varepsilon) + \frac{(\psi,\eta)^2}{(\eta,\eta)^2}(\eta,\eta) + \frac{(\psi,\zeta)^2}{(\zeta,\zeta)^2}(\zeta,\zeta) + \frac{237}{(\psi,\psi_0)^2}$$

$= \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + (\psi,\psi_0)^2$ giving

$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 < 1$.

As in Step 9, $(\varepsilon,\varepsilon) = p(p-1)/2$, $(\eta,\eta) = \frac{q(q-1)}{2}$

$(\zeta,\zeta) = \frac{r(r-1)}{2}$. Then from $(\psi,\varepsilon) = (p-1)(\psi,\varepsilon_{p-1})$,

$$\cos^2\theta_1 = \frac{(\psi,\varepsilon)^2}{(\varepsilon,\varepsilon)} = \frac{(p-1)^2(\psi,\varepsilon_{p-1})^2}{p(p-1)/2} = \frac{2(p-1)\frac{1}{4}}{p} =$$

$\frac{1}{2}\left(1-\frac{1}{p}\right)$ since $1 = \langle\psi,\varepsilon_{p-1}\rangle\langle\varepsilon_{p-1},\psi\rangle =$

$$\frac{4(\psi,\varepsilon_{p-1})^2}{(\psi,\psi)(\varepsilon_{p-1},\varepsilon_{p-1})} = 4(\psi,\varepsilon_{p-1})^2.$$

Similarly, $\cos^2\theta_2 = \frac{1}{2}(1-\frac{1}{q})$ and $\cos^2\theta_3 = \frac{1}{2}(1-\frac{1}{r})$, so we get $\frac{1}{2}(1-\frac{1}{p}+1-\frac{1}{q}+1-\frac{1}{r}) < 1$ which means $1 < \frac{1}{p}+\frac{1}{q}+\frac{1}{r}$. If necessary, by changing the labels of the unit vectors, we can assume that $\frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{r} \leq \frac{1}{2}$ since if $r=1$ (or $p=1$ or $q=1$) then the graph is A_n type (simple chain) without a triple node. Now we have $1 < \frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq \frac{1}{r}+\frac{1}{r}+\frac{1}{r} = \frac{3}{r} \leq \frac{3}{2}$ so $2 \leq r < 3$ forces $r=2$.

This gives $\frac{1}{2} < \frac{1}{p} + \frac{1}{q} \leq \frac{1}{q} + \frac{1}{q} = \frac{2}{q} \leq \frac{2}{2} = 1$

so $2 \leq q < 4$. If $q = 3$ then $\frac{1}{6} < \frac{1}{p}$ so $p < 6$ and $\frac{1}{p} \leq \frac{1}{3}$ says $3 \leq p < 6$.

If $q = 2$ then $1 < \frac{1}{p} + \frac{1}{2} + \frac{1}{2}$ just says $0 < \frac{1}{p}$ is no restriction on p.

The options for triples (p, q, r) are then:

$(p, 2, 2) \leftrightarrow D_n$ or $(3, 3, 2) \leftrightarrow E_6$ or $(4, 3, 2) \leftrightarrow E_7$ or $(5, 3, 2) \leftrightarrow E_8$.

So connected graphs Γ of admissible sets in E are included in the Coxeter graphs of types A–G in the Table on page 58.

In all cases except B_ℓ and C_ℓ, the
Coxeter graph uniquely determines the Dynkin diagram. The choice of arrow direction for the double edge distinguishes B_ℓ from C_ℓ. $\square$

Construction of root systems:

While all possible connected Dynkin diagrams of irred. root systems were classified in the theorem just proved, the existence of all the listed root systems remains to be seen. Here we will explicitly construct each type of root system and also get the structure of the Weyl groups.

Let $\mathbb{R}^n$ have std. dot product, $\{\varepsilon_1, \dots, \varepsilon_n\}$ the std. orthonormal basis. Let I be the the lattice $\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. We will define Φ in each case as the set of all $v \in I$ (or a closely related J) with $\|v\|$ in a specified set.

I and J are discrete in the usual $\mathbb{R}^n$ topology, and $\{v \in \mathbb{R}^n \mid \|v\| \in \{c_1, c_2\}\}$ for fixed constants c_1 and c_2 (if needed) is closed and bounded so compact. So that implies Φ is a finite set, and we will always have $0 \notin \Phi$. Our choice of Φ will be easily seen to span E, where either $E = \mathbb{R}^n$ or a subspace of $\mathbb{R}^n$. Root system axiom (R2) will be clear from the choice of c_1 and c_2. To check axiom (R3), it will be enough to check that $\sigma_\alpha(\Phi) \subseteq J$ since σ_α will not change lengths. For $\alpha, \beta \in I$, we will easily have $(\alpha, \beta) \in \mathbb{Z}$ so $\langle \alpha, \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$ if $2/(\beta, \beta) \in \mathbb{Z}$.

A_ℓ $(\ell \geq 1)$: Let $E = \{v \in \mathbb{R}^{\ell+1} \mid \underbrace{(v, \varepsilon_1 + \cdots + \varepsilon_{\ell+1})}_{=0}\}$ $= (\varepsilon_1 + \cdots + \varepsilon_{\ell+1})^\perp \leq \mathbb{R}^{\ell+1}$. $I = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_{\ell+1}$, $I' = I \cap E$, $\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 2\}$. So $\alpha \in \Phi$ iff $\alpha = \sum_{i=1}^{\ell+1} a_i \varepsilon_i$ with $0 = (\alpha, \varepsilon_1 + \cdots + \varepsilon_{\ell+1}) = \sum_{i=1}^{\ell+1} a_i$ and $2 = (\alpha, \alpha) = \sum_{i=1}^{\ell+1} a_i^2$. The second condition means exactly two coefficients are nonzero, with each of abs. value 1, and the first condition means the two coefficients add up to 0. So $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq \ell+1\}$. Letting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq \ell$ we see $\{\alpha_1, \ldots, \alpha_\ell\}$ is indep. and for $i < j$, $\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$.

So $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a base for Φ. The Cartan matrix $A = [\langle \alpha_i, \alpha_j \rangle]$ is the type A_ℓ matrix,

$$\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i, \alpha_j) = (\varepsilon_i - \varepsilon_{i+1}, \varepsilon_j - \varepsilon_{j+1})$$

$$= \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i - j = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, we can compute the effect of reflection σ_{α_i} on ε_j as follows:

$$\sigma_{\alpha_i}(\varepsilon_j) = \varepsilon_j - \langle \varepsilon_j, \alpha_i \rangle \alpha_i = \varepsilon_j - (\varepsilon_j, \varepsilon_i - \varepsilon_{i+1})(\varepsilon_i - \varepsilon_{i+1})$$

$$= \varepsilon_j - (\delta_{ij} - \delta_{i+1,j})(\varepsilon_i - \varepsilon_{i+1}) = \begin{cases} \varepsilon_{i+1} & \text{if } i = j. \\ \varepsilon_i & \text{if } i+1 = j. \\ \varepsilon_j & \text{if } j \notin \{i, i+1\} \end{cases}$$

so σ_{α_i} interchanges $\varepsilon_i \leftrightarrow \varepsilon_{i+1}$ and fixes all ε_j for $j \notin \{i, i+1\}$.

This means σ_{α_i} acts on the subscripts of any linear combination of basis $\{\varepsilon_1, \dots, \varepsilon_{\ell+1}\}$ as transposition $(i, i+1)$. Those generate the permutation group $S_{\ell+1} \cong W$.

B_ℓ $(\ell \geq 2)$. Let $E = \mathbb{R}^\ell$, $\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 1, 2\}$, $I = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_\ell$, so as above
$\Phi = \{\pm\varepsilon_i \mid 1 \leq i \leq \ell\} \cup \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq \ell\}$.
Letting $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell$, $\alpha_\ell = \varepsilon_\ell$ we see $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ is indep., spans E, and short pos. root $\varepsilon_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{\ell-1} + \alpha_\ell$, long pos. root $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$ for $i < j$ and