$\varepsilon_i + \varepsilon_j = (\alpha_i + \alpha_{i+1} + \cdots + \alpha_\ell) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_\ell)$
for $i < j$. The Cartan matrix $A = [\langle \alpha_i, \alpha_j \rangle]$
has $\langle \alpha_i, \alpha_j \rangle = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{cases} (\alpha_i, \alpha_j) & \text{for } 1 \le i, j \le \ell - 1 \\ 2(\alpha_i, \varepsilon_\ell) & \text{for } 1 \le i \le \ell,\ j = \ell \\ \underbrace{(\varepsilon_\ell, \alpha_{\ell-1})}_{= -1} & \text{for } i = \ell,\ j = \ell - 1 \end{cases}$

so get type B_ℓ matrix.

The Weyl group contains σ_{ε_i} as well as $\sigma_{\varepsilon_i - \varepsilon_j}$, so it contains permutations of $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$ as well as sign changes of each ε_i. This gives $W \cong (\mathbb{Z}_2)^\ell \rtimes S_\ell$ semidirect product.

C_ℓ $(\ell \geq 3)$: $E = \mathbb{R}^\ell$,
$\Phi = \{\pm 2\varepsilon_i \mid 1 \leq i \leq \ell\} \cup \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq \ell\}$,
with base $\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \alpha_\ell = 2\varepsilon_\ell\}$
and W the same as for type B_ℓ.

D_ℓ $(\ell \geq 4)$: $E = \mathbb{R}^\ell$, $\Phi = \{\alpha \in I \mid (\alpha, \alpha) = 2\} =$
$\{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq \ell\}$, with base
$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell\}$
gives D_ℓ Cartan matrix (Dynkin diagram).
Weyl gp contains permutations S_ℓ but only even sign changes, so $W \cong \mathbb{Z}_2^{\ell-1} \rtimes S_\ell$.

$E_6 \subset E_7 \subset E_8$ so suffices to construct Φ of type E_8. This is interesting, so let's come back to it when we have more time.

F_4: $E = \mathbb{R}^4$, $I' = I + \mathbb{Z}\left(\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}\right)$,

$\Phi = \{\alpha \in I' \mid (\alpha,\alpha) = 1, 2\} = \{\pm\varepsilon_i \mid 1 \le i \le 4\} \cup \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le 4\} \cup \left\{\pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \;\middle|\; \begin{array}{l}\text{all choices}\\ \text{of } \pm \text{ ok}\end{array}\right\}$

$\Delta = \{\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$ gives F_4 Dynkin diagram. $|W| = 1152$.

G_2: $E = \{v \in \mathbb{R}^3 \mid (v, \varepsilon_1+\varepsilon_2+\varepsilon_3) = 0\}$, $I' = I \cap E$,

$\Phi = \{\alpha \in I' \mid (\alpha,\alpha) = 2, 6\} = \{\pm(\varepsilon_1-\varepsilon_2), \pm(\varepsilon_2-\varepsilon_3), \pm(\varepsilon_1-\varepsilon_3), \pm(2\varepsilon_1-\varepsilon_2-\varepsilon_3), \pm(2\varepsilon_2-\varepsilon_1-\varepsilon_3), \pm(2\varepsilon_3-\varepsilon_1-\varepsilon_2)\}$. $\Delta = \left\{\begin{array}{l}\alpha_1 = \varepsilon_1 - \varepsilon_2,\\ \alpha_2 = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1\end{array}\right\}$

E_8: Let $E = \mathbb{R}^8$, $I = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_8$,

$I' = I + \mathbb{Z}(\varepsilon_1 + \cdots + \varepsilon_8)/2$ and let

$$I'' = \left\{ \sum_{i=1}^{8} c_i \varepsilon_i + c(\varepsilon_1 + \cdots + \varepsilon_8)/2 \;\middle|\; c_i, c \in \mathbb{Z}, \sum_{i=1}^{8} c_i \in 2\mathbb{Z} \right\}.$$

Then $\Phi = \{\alpha \in I'' \mid (\alpha, \alpha) = 2\}$ certainly includes

$\{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le 8\}$ (a rootsys. of type D_8)

and also $\left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{k_i} \varepsilon_i \;\middle|\; k_i = 0, 1 \text{ and } \sum_{i=1}^{8} k_i \in 2\mathbb{Z} \right\}$.

<u>Note</u>: For $\alpha = \frac{1}{2}(\pm 1, \pm 1, \dots, \pm 1) \in \frac{1}{2}\mathbb{Z}^8$, $(\alpha, \alpha) = \frac{8}{4} = 2$.

Using Clifford algebras we can construct two irreducible D_8 modules ($so(16, F)$) whose weights are of this form, one with $\sum k_i$ <u>even</u>, the other <u>odd</u>.

<u>Note</u>: The root system of type D_8 has $4\binom{8}{2} = 4\cdot\frac{8\cdot 7}{2} = 4(28) = 112$ roots. From p. 3 of Humphreys, $\dim(\mathfrak{o}(2\ell,F)) = 2\ell^2-\ell$ so $\dim(\mathfrak{o}(16,F)) = 2(8^2)-8 = 2(64)-8 = 120$. $\mathfrak{o}(16,F) = H \oplus \bigoplus_{\alpha\in\Phi(D_8)} L_\alpha$ with $\dim(H) = 8$ and $\dim(L_\alpha) = 1$, so $|\Phi(D_8)| = 120-8 = 112$.

The number of vectors in $\left\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{k_i}\varepsilon_i \,\middle|\, k_i = 0,1\right\}$ is $2^8 = 256$, half have $\sum k_i$ even and half have $\sum k_i$ odd, so each half has $2^7 = 128$ vectors. Then the two sets in $\Phi(E_8)$ together contain $112+128 = 240$ vectors. Rank$(E_8) = 8$ so a Lie alg. of type E_8 would have dim. 248.

<u>Exercise</u>: Check that all inner products between vectors in $\Phi(E_8)$, the 240 vectors from the two explicitly given sets on p. 249, are in $\mathbb{Z}$.

With base $\Delta = \{\alpha_1 = \frac{1}{2}(\varepsilon_1 - (\varepsilon_2 + \cdots + \varepsilon_7) + \varepsilon_8), \alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = \varepsilon_2 - \varepsilon_1, \alpha_4 = \varepsilon_3 - \varepsilon_2, \alpha_5 = \varepsilon_4 - \varepsilon_3, \alpha_6 = \varepsilon_5 - \varepsilon_4, \alpha_7 = \varepsilon_6 - \varepsilon_5, \alpha_8 = \varepsilon_7 - \varepsilon_6\}$ we get the Dynkin diagram of E_8 as in Table 1 on p. 58.

The E_8 Weyl group has order $2^{14} 3^5 5^2 7$ $= 4!\,6!\,8! = 696{,}729{,}600$

For more on Wikipedia see the page "E8 lattice"

Automorphisms of rootsystems:

The Lemma on p. 43 of Humphreys implies that $W \trianglelefteq \mathrm{Aut}(\Phi)$. Let $\Gamma = \{\sigma \in \mathrm{Aut}(\Phi) \mid \sigma(\Delta) = \Delta\}$ for fixed base Δ of Φ, so $\Gamma \leq \mathrm{Aut}(\Phi)$ and $\Gamma \cap W = \{1\}$ since W action on bases is simply transitive. $\forall \tau \in \mathrm{Aut}(\Phi)$, $\tau(\Delta) = \Delta'$ is another base of Φ, so $\exists \sigma \in W$ s.t. $\sigma\tau(\Delta) = \Delta$ so $\sigma\tau \in \Gamma$ and $\tau \in W\Gamma = \Gamma W$. This implies that $\mathrm{Aut}(\Phi) = W \rtimes \Gamma$ is a semidirect product.

$\forall \tau \in \mathrm{Aut}(\Phi)$, $\forall \alpha, \beta \in \Phi$, have $\langle \tau\alpha, \tau\beta \rangle = \langle \alpha, \beta \rangle$ so τ gives an automorphism (symmetry) of the Dynkin diagram of Φ. A trivial action of τ

on the diagram means $\tau = 1$ since Δ is a basis of E. By Prop. on p. 55, every Dynkin diagram symmetry gives a $\tau \in \mathrm{Aut}(\Phi)$, but diagram symmetries preserve Δ, so they correspond to elements of Γ, really to cosets in $\mathrm{Aut}(\Phi)/W \cong \Gamma \leftrightarrow \{\text{diagram symmetries}\}$.

See Table 1 on p. 66 for Γ and other useful information about each type of irred. Φ. Table 2 on p. 66 contains highest long and short (if any) roots for each type of Φ.