Abstract Theory of Weights: (base Δ)

Let Φ be a root system in E with Weyl gp. W.

Def. Let $\Lambda = \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Phi\}$ (weight lattice)

Then Λ is a subgroup of E under $+$ since $\langle \lambda, \alpha \rangle$ is a linear function of λ. Clearly, $\Phi \subseteq \Lambda$, and $\lambda \in \Lambda$ iff $\langle \lambda, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta$.

Pf. ($\Rightarrow$) Obvious. ($\Leftarrow$) $\forall \sigma \in W$, $\langle \lambda, \alpha \rangle = \langle \sigma\lambda, \sigma\alpha \rangle$ and $W(\Phi) = \Phi$ so $\sigma\lambda \in \Lambda$ and then $W(\Lambda) = \Lambda$. $\forall \alpha \in \Phi$, $\exists \sigma \in W$ s.t. $\sigma\alpha \in \Delta$ so if $\langle \Lambda, \Delta \rangle \in \mathbb{Z}$ then $\langle W(\Lambda), W(\Delta) \rangle = \langle \Lambda, \Phi \rangle \in \mathbb{Z}$. $\square$

Def. Let $\Lambda_r = \mathbb{Z}\Delta = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_\ell$ be the root lattice, the subgroup of Λ generated by Φ (or by Δ).

<u>Def</u>. Let $\Lambda^+ = \{\lambda \in \Lambda \mid 0 \le \langle \lambda, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta\}$ be the set of dominant (integral) weights, and $\Lambda^{++} = \{\lambda \in \Lambda \mid 0 < \langle \lambda, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta\}$ be the set of strictly (strongly) dominant weights.

So $\Lambda^+ = \Lambda \cap \overline{C(\Delta)}$ and $\Lambda^{++} = \Lambda \cap C(\Delta)$.

Since $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a basis of E, so is

$$\Delta^\vee = \left\{ \frac{2\alpha_i}{(\alpha_i, \alpha_i)} = \alpha_i^\vee \;\middle|\; 1 \le i \le \ell \right\}.$$

<u>Def</u>. Let $\{\lambda_1, \dots, \lambda_\ell\} \subseteq E$ be the dual basis to $\Delta^\vee$, that is, s.t.

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \langle \lambda_i, \alpha_j \rangle = \delta_{ij}.$$

Then $\lambda_i \in \Lambda^+$, and these are called the <u>fundamental</u> (dominant) <u>weights</u> (w.r.t. Δ).

Note. Let $\sigma_i = \sigma_{\alpha_i}$ for $1 \le i \le \ell$ and check $\sigma_i(\lambda_j) = \lambda_j - \langle \lambda_j, \alpha_i \rangle \alpha_i = \lambda_j - \delta_{ij}\alpha_i$ for $1 \le i,j \le \ell$.

$\forall \lambda \in E$ let $m_i = \langle \lambda, \alpha_i \rangle$ so $0 = \langle \lambda - \sum_{i=1}^{\ell} m_i \lambda_i, \alpha_j \rangle$ for all $1 \le j \le \ell$ so $0 = (\lambda - \sum m_i \lambda_i, \alpha_j)$ giving $\lambda = \sum m_i \lambda_i$.

This shows that $\Lambda = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_\ell$ is a lattice ($\mathbb{Z}$-span of a basis of E), and $\Lambda^+ = \mathbb{Z}^+\lambda_1 + \cdots + \mathbb{Z}^+\lambda_\ell$ for $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \ge 0\}$.

Ex. For $\Delta = \{\alpha_1, \alpha_2\}$ of type A_2 with Cartan matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ find fund. wts. $\{\lambda_1, \lambda_2\}$.

Solution: $\alpha_1 = \langle \alpha_1, \alpha_1 \rangle \lambda_1 + \langle \alpha_1, \alpha_2 \rangle \lambda_2 = 2\lambda_1 - \lambda_2$

$\alpha_2 = \langle \alpha_2, \alpha_1 \rangle \lambda_1 + \langle \alpha_2, \alpha_2 \rangle \lambda_2 = -\lambda_1 + 2\lambda_2$

shows that the Cartan matrix directly gives simple roots as integral lin. combinations of the fund. wts. So the inverse of the Cartan matrix expresses the fund. wts. as lin. combos of the roots, and the coefficients will be rationals with denominators at most $\det(A)$.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ so } \begin{aligned} \lambda_1 &= \tfrac{1}{3}(2\alpha_1 + \alpha_2) \\ \lambda_2 &= \tfrac{1}{3}(\alpha_1 + 2\alpha_2). \end{aligned}$$

See Fig. 1 on p. 68.

<u>Th</u>: Λ/Λ_r is a finite group, called the fundamental group of Φ.

<u>Pf</u>. Write $\alpha_i = \sum_{j=1}^{\ell} m_{ij}\lambda_j$ where $m_{ij} = \langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$ so $A = [m_{ij}]$ is the Cartan matrix of Δ.

The transition matrix from the simple roots basis Δ to the fund. wts. basis $\{\lambda_1, \dots, \lambda_\ell\} =: F$ is the matrix ${}_F P_\Delta$ s.t. $Col_j({}_F P_\Delta) = [\alpha_j]_F = \begin{bmatrix} m_{j1} \\ \vdots \\ m_{j\ell} \end{bmatrix}$ so ${}_F P_\Delta = \begin{bmatrix} m_{11} & m_{21} & \cdots & m_{\ell 1} \\ \vdots & \vdots & & \vdots \\ m_{1\ell} & m_{2\ell} & \cdots & m_{\ell\ell} \end{bmatrix} = [m_{ji}] = A^{Tr}$

is the transpose of the Cartan matrix.

Then ${}_\Delta P_F = {}_F P_\Delta^{-1} = (A^{-1})^{Tr} = (A^{Tr})^{-1}$ has rational entries with denominators $\det(A^{Tr}) = \det(A)$.

Let ${}_\Delta P_F = [n_{ij}/\det(A)]$ for $n_{ij} \in \mathbb{Z}$, so

$\lambda_j = \sum_{i=1}^{\ell} \frac{n_{ij}}{\det A} \alpha_i$ says $\det(A)\lambda_j \in \Lambda_r$ and

$|\Lambda/\Lambda_r| = \det(A)$. □

In the A_2 type $\det(A) = 3$ and you can use the formulas for λ_1 and λ_2 to find the 3 cosets of Λ/Λ_r. For example, $\lambda_1 + \lambda_2 = \frac{1}{3}(3\alpha_1 + 3\alpha_2) = \alpha_1 + \alpha_2 \in \Lambda_r$ and, of course, $3\lambda_1 = 2\alpha_1 + \alpha_2 \in \Lambda_r$ and $3\lambda_2 = \alpha_1 + 2\alpha_2 \in \Lambda_r$ and $2\lambda_1 - \lambda_2 = \alpha_1 \in \Lambda_r$ and $-\lambda_1 + 2\lambda_2 = \alpha_2 \in \Lambda_r$.

These mean that $2\lambda_1 + \Lambda_r = \lambda_2 + \Lambda_r$ and $2\lambda_2 + \Lambda_r = \lambda_1 + \Lambda_r$. So among the cosets

$$\{\Lambda_r, \lambda_1 + \Lambda_r, 2\lambda_1 + \Lambda_r, \lambda_2 + \Lambda_r, \lambda_1 + \lambda_2 + \Lambda_r, \lambda_2 + 2\lambda_1 + \Lambda_r, 2\lambda_2 + \Lambda_r, \lambda_1 + 2\lambda_2 + \Lambda_r, 2\lambda_1 + 2\lambda_2 + \Lambda_r\}$$

there are only 3 distinct cosets:

$\Lambda_r = \lambda_1 + \lambda_2 + \Lambda_r = 2\lambda_1 + 2\lambda_2 + \Lambda_r$

$\lambda_1 + \Lambda_r = 2\lambda_2 + \Lambda_r = 2\lambda_1 + \lambda_2 + \Lambda_r$

$2\lambda_1 + \Lambda_r = \lambda_2 + \Lambda_r = \lambda_1 + 2\lambda_2 + \Lambda_r$

<u>Th</u>: For irreducible types of Φ we have:

Type	A_ℓ	B_ℓ	C_ℓ	D_ℓ	E_6	E_7	E_8	F_4	G_2
$\lvert\Lambda/\Lambda_r\rvert$	$\ell+1$	2	2	4	3	2	1	1	1

Explicit formulas for each type expressing λ_i in terms of α_j's are in Table 1 on p. 69. These imply the group structure of Λ/Λ_r.

<u>Lemma A</u>. $\forall \lambda \in \Lambda$, $\exists! \, \sigma\lambda \in \Lambda^+$ for some $\sigma \in W$. If $\lambda \in \Lambda^+$ then $\forall \sigma \in W$, $\sigma\lambda \leq \lambda$. If $\lambda \in \Lambda^{++}$ then $\sigma\lambda = \lambda$ iff $\sigma = 1$.

<u>Lemma B.</u> For $\lambda \in \Lambda^+$, $\{\mu \in \Lambda^+ \mid \mu \leq \lambda\}$ is a finite set.

<u>Pf.</u> From $\lambda, \mu \in \Lambda^+$ we get $\lambda + \mu \in \Lambda^+$. From $\mu \leq \lambda$ we get $\lambda - \mu = \sum_{i=1}^{\ell} k_i \alpha_i$ for $0 \leq k_i \in \mathbb{Z}$.

So $(\lambda+\mu, \lambda-\mu) = (\lambda+\mu, \sum k_i \alpha_i) = \sum k_i (\lambda+\mu, \alpha_i) \geq 0$
$= (\lambda,\lambda) - (\mu,\mu)$ so $(\mu,\mu) \leq (\lambda,\lambda)$. The compact set (ball), $B_\lambda = \{x \in E \mid (x,x) \leq (\lambda,\lambda)\}$ is a ball of radius $\sqrt{(\lambda,\lambda)}$ centered at the origin of E.

$\{\mu \in \Lambda^+ \mid \mu \leq \lambda\} = B_\lambda \cap \Lambda^+$ is the intersection of a compact set and a discrete set, so it must be a finite set. $\square$

Recall we defined $\delta = \frac{1}{2} \sum_{0 < \alpha \in \Phi} \alpha$ and showed

$\sigma_i(\delta) = \delta - \alpha_i$ for $1 \leq i \leq l$.

<u>Lemma C.</u> $\delta = \sum_{j=1}^{l} \lambda_j \in \Lambda^{++}$.

<u>Pf.</u> $\sigma_i(\delta) = \delta - \alpha_i = \delta - \langle \delta, \alpha_i \rangle \alpha_i$ shows $\langle \delta, \alpha_i \rangle = 1$ for $1 \leq i \leq l$, so $\delta = \sum_{j=1}^{l} \langle \delta, \alpha_j \rangle \lambda_j = \sum_{j=1}^{l} \lambda_j$. $\square$

<u>Lemma D.</u> Let $\mu \in \Lambda^{+}$ and $\nu = \sigma^{-1}\mu$ for $\sigma \in W$. Then $(\nu+\delta, \nu+\delta) \leq (\mu+\delta, \mu+\delta)$ with "=" iff $\nu = \mu$.

<u>Pf.</u> We have $(\nu+\delta, \nu+\delta) = (\sigma(\nu+\delta), \sigma(\nu+\delta)) = (\mu+\sigma\delta, \mu+\sigma\delta) = (\mu,\mu) + 2(\mu, \sigma\delta) + (\delta,\delta) =$

$(\mu+\delta, \mu+\delta) - 2(\mu, \delta - \sigma\delta) = (\mu,\mu) + 2(\mu,\delta) + (\delta,\delta)$
$- 2(\mu,\delta) + 2(\mu,\sigma\delta)$.