Since $\mu \in \Lambda^+$ and $\delta - \sigma\delta = \sum_{i=1}^{\ell} k_i \alpha_i$ for $0 \le k_i \in \mathbb{Z}$, $(\mu, \delta - \sigma\delta) \ge 0$ so $(\nu+\delta, \nu+\delta) \le (\mu+\delta, \mu+\delta)$ with "=" iff $(\mu, \delta - \sigma\delta) = 0$ iff $(\mu, \delta) = (\mu, \sigma\delta) = (\nu, \delta)$ iff $(\mu - \nu, \delta) = 0$. But $\nu \le \mu$ from Lemma A, so $\mu - \nu = \sum_{i=1}^{\ell} n_i \alpha_i$ for $0 \le n_i \in \mathbb{Z}$ so from Lemma C,

$$(\mu-\nu, \delta) = \sum_{i=1}^{\ell} n_i \sum_{j=1}^{\ell} (\alpha_i, \lambda_j) = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} n_i (\lambda_j, \alpha_i) =$$

$$\sum_{i=1}^{\ell} \sum_{j=1}^{\ell} n_i \frac{(\alpha_i, \alpha_i)}{2} \underbrace{\langle \lambda_j, \alpha_i \rangle}_{=\delta_{ij}} = \sum_{i=1}^{\ell} n_i \frac{(\alpha_i, \alpha_i)}{2} = 0 \text{ iff}$$

all $n_i = 0$ iff $\mu = \nu$. $\square$

Saturated sets of weights:

Def. Say that a subset $\Pi \subseteq \Lambda$ is *saturated* if $\forall \lambda \in \Pi$, $\forall \alpha \in \Phi$, for all $0 \leq i \leq \langle \lambda, \alpha \rangle$, or $\langle \lambda, \alpha \rangle \leq i \leq 0$, $\lambda - i\alpha \in \Pi$.

Note: Π saturated implies $W(\Pi) \subseteq \Pi$ since $\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha \in \Pi$ and $W = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle$.

Def. Say saturated set Π has *highest wt.* λ (which must be in Λ^+) if $\lambda \in \Pi$ and $\forall \mu \in \Pi$, $\mu \leq \lambda$.

Examples: $\{0\}$ is saturated with highest wt. 0.
$\Phi \cup \{0\}$ for root system Φ of a semisimple Lie alg.

If Φ is irred. and Δ is a base of Φ giving a partial order $\leq$, then Φ has a unique highest root and that is the highest wt. of $\Pi = \Phi \cup \{0\}$.

Lemma E. A saturated set with a highest weight λ must be a finite set.

Pf. Let Π be saturated, $\lambda \in \Pi$ highest wt. From Lemma B, $\{\mu \in \Lambda^+ \mid \mu \leq \lambda\}$ is finite so $\Pi^+ = \{\mu \in \Pi \cap \Lambda^+ \mid \mu \leq \lambda\}$ is finite. $|W|$ is finite so $|W(\Pi^+)|$ is finite. Π is W-invariant and each W orbit in Π includes an element $\mu \in \Lambda^+ \cap \Pi = \Pi^+$ so $\Pi = W(\Pi^+)$ is finite. $\square$

Lemma F. Let Π be saturated with highest wt. λ. If $\mu \in \Lambda^+$ and $\mu \le \lambda$ then $\mu \in \Pi$.

Pf. Suppose $\mu' = \mu + \sum_{\alpha \in \Delta} k_\alpha \alpha \in \Pi$ for $k_\alpha \in \mathbb{Z}^+$.

Note: μ' need not be in Λ^+. We will show how to reduce a k_α by 1 and stay in Π. Repeating the process eventually gives $\mu' = \mu \in \Pi$. The base case of this process is when $\mu' = \lambda$.

If $\mu' \neq \mu$ then some $k_\alpha > 0$ and $\left(\sum k_\alpha \alpha, \sum k_\alpha \alpha\right) > 0$ so $\exists \beta \in \Delta$ s.t. $\left(\sum k_\alpha \alpha, \beta\right) > 0$ and $k_\beta > 0$.

So $\langle \sum k_\alpha \alpha, \beta \rangle > 0$. Also, $\mu \in \Lambda^+$ so $\langle \mu, \beta \rangle \geq 0$ and then $\langle \mu', \beta \rangle > 0$. Π saturated, so $\mu' - \beta \in \Pi$ giving a new μ' and an expression for it with $k_\beta - 1$ in place of k_β. $\square$

Thus, saturated set Π with highest wt. λ is exactly $W\{\mu \in \Lambda^+ \mid \mu \leq \lambda\}$ which is uniquely determined by $\lambda \in \Lambda^+$, so could be denoted by Π^λ.

<u>Lemma G.</u> Let Π^λ be a saturated set with highest wt. $\lambda \in \Lambda^+$. $\forall \mu \in \Pi^\lambda$ we have $(\mu+\delta, \mu+\delta) \leq (\lambda+\delta, \lambda+\delta)$ with "=" iff $\mu = \lambda$.

<u>Pf.</u> From Lemma D, it suffices to prove this for $\mu \in \Lambda^+$. Write $\mu = \lambda - \eta$ for $\eta = \sum_{i=1}^{\ell} n_i \cdot \alpha_i$ with $0 \leq n_i \in \mathbb{Z}$. Then

$$(\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta) = (\lambda+\delta, \lambda+\delta) - (\lambda+\delta-\eta, \lambda+\delta-\eta)$$
$$= (\lambda+\delta, \eta) + (\eta, \lambda+\delta-\eta) = (\lambda+\delta, \eta) + (\eta, \mu+\delta) \geq (\lambda+\delta, \eta) \geq 0$$

where we used $\mu+\delta \in \Lambda^{++}$ and $\lambda+\delta \in \Lambda^{++}$. $\boxed{268}$

So only get "=" 0 when $\eta = 0$. $\square$

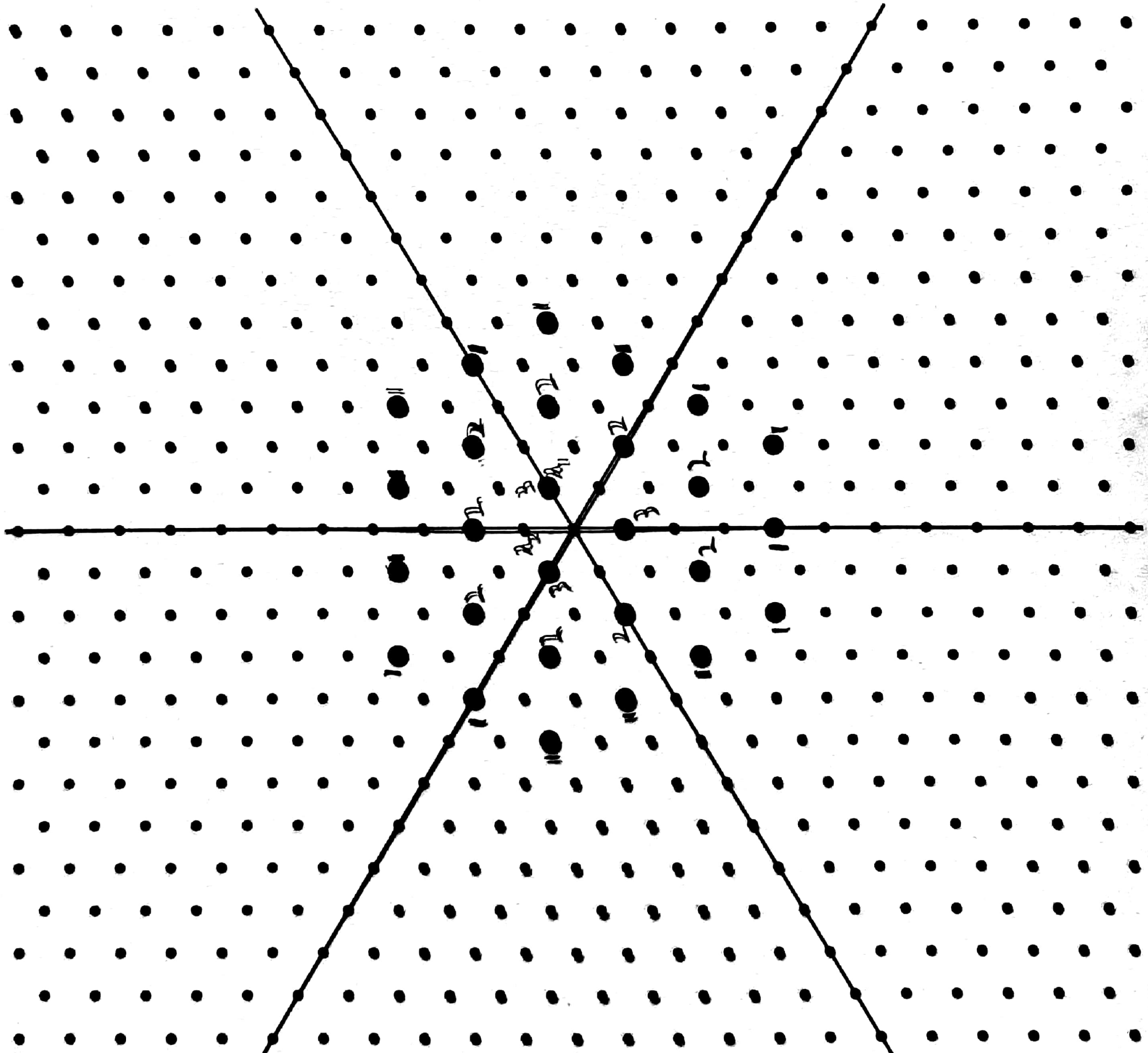

Figure 8: A_2 Weight Diagram For Irreducible Module With Highest Weight $3\lambda_1 + 2\lambda_2$