Clifford Algebras: $\mathrm{Char}(F) \neq 2$

Assume V is a fin. dim'l vector space over field F, and $(\cdot,\cdot): V \times V \to F$ is a symmetric bilinear form. Let $\mathrm{Cliff}(V,(\cdot,\cdot))$ be the associate algebra with unit element 1, generated by V s.t. $v_1 v_2 + v_2 v_1 = (v_1, v_2)1$.

If V has a basis $B = \{a_1, \dots, a_m, a_1^*, \dots, a_m^*\}$ and the matrix of the form w.r.t. B is $J = \left[\begin{array}{c|c} 0 & I_m \\ \hline I_m & 0 \end{array}\right]$

so $(a_i, a_j) = 0 = (a_i^*, a_j^*)$, $(a_i, a_j^*) = \delta_{ij}$ for $1 \le i, j \le m$. So

$a_i a_i + a_i a_i = 0$ says $a_i^2 = 0$ and

$a_i^* a_i^* + a_i^* a_i^* = 0$ says $(a_i^*)^2 = 0$ but

$a_i a_j^* + a_j^* a_i = \delta_{ij}$ and

$a_i a_j + a_j a_i = 0$ so $a_i a_j = -a_j a_i$ and

$a_i^* a_j^* + a_j^* a_i^* = 0$ so $a_i^* a_j^* = -a_j^* a_i^*$.

<u>Th</u>: A basis of Cliff_m is

$$\{(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} a_1^{l_1} \cdots a_m^{l_m} 1 \mid k_i, l_i = 0, 1\}$$

so $\dim(\mathrm{Cliff}_m) = 2^{2m}$.

Let $\mathcal{I}_m$ be the left ideal of Cliff_m gen. by $a_1, \ldots, a_m$, so the basis vectors in $\mathcal{I}_m$ are those with $l_i \neq 0$ for some $1 \leq i \leq m$.

Let $CM_m = \mathrm{Cliff}_m / \mathcal{I}_m$ which has basis

$\{(a_1^*)^{k_1}\cdots(a_m^*)^{k_m}\,\mathcal{I} \mid k_i = 0,1\}$ so

$\dim(CM_m) = 2^m$.

Have a representation (Clifford module) of Cliff_m on CM_m by left multiplication, where a_i^* are "creation" operators, and a_i are "annihilation" operators, $1+\mathcal{I}_m \in CM_m$ is the "vacuum" vector s.t. $a_i\cdot(1+\mathcal{I}_m) = 0+\mathcal{I}_m$. Call these operators "fermionic" because $a_i^2 = 0 = (a_i^*)^2$ and they anti-commute.

How does a_i act on a basis vector of CM_m?

$$a_i\,(a_1^*)^{k_1}\cdots(a_i^*)^{k_i}\cdots(a_m^*)^{k_m}] + I_m$$

$$= (-1)^{k_1+\cdots+k_{i-1}}\,\underbrace{(a_1^*)^{k_1}\cdots(a_{i-1}^*)^{k_{i-1}}}_{''}\,a_i\,(a_i^*)^{k_i}\cdots(a_m^*)^{k_m}]$$

$$= \quad '' \quad \big((-a_i^*)^{k_i}a_i+\delta_{i,i}\big)\cdots(a_m^*)^{k_m}]$$

if $k_i=1$

$$= \begin{cases} 0 & \text{if } k_i=0 \\ (-1)^{k_1+\cdots+k_{i-1}}(a_1^*)^{k_1}\cdots(a_{i-1}^*)^{k_{i-1}}(a_i^*)^{0}\cdots(a_m^*)^{k_m}] & \text{if } k_i=1 \end{cases}$$

so left mult. by a_i either kills the basis vector if $k_i=0$, or changes $k_i=1$ to $k_i=0$ (with a possible sign change).

Th. CM_m is an irred. $Cliff_m$-module.

Pf. Use annihilation operators a_i to remove creation op's from 1, get just 1, from which CM_m can be reached.

Next step: Define a subspace of $Cliff_m$ which forms a Lie algebra under commutation.

Def. Let U be the subspace of $Cliff_m$ spanned by $\{a_i, a_i^* \mid 1 \le i \le m\}$. $\forall\, a,b \in U$ let $:ab: = \frac{1}{2}(ab-ba)$ $= -:ba:$
and let $\mathfrak{g} = \mathrm{span}\{:ab: \mid a,b \in U\}$.

Since $ab+ba = (a,b)1$ we have

$$:ab: = \frac{1}{2}(ab-ba) = \frac{1}{2}(ab+ab-(a,b)1) = ab - \frac{1}{2}(a,b)1$$

$$= \frac{1}{2}(-ba+(a,b)1-ba) = -ba + \frac{1}{2}(a,b)1.$$